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Chapter 1

Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets

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Abstract. This work investigates the evolution of traditional set theory to address complex and ambiguous real-world phenomena. It introduces hierarchical hyperstructures and superhyperstructures, where superhyperstructures are formed by iteratively applying power sets to create nested abstractions.

The focus is placed on three foundational set-based frameworks—Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets—and their extensions into Hyperfuzzy Sets, HyperNeutrosophic Sets, and Hyperplithogenic Sets. These extensions are applied to various domains, including Statistics, TOPSIS, K-means Clustering, Evolutionary Theory, Topological Spaces, Decision Making, Probability, and Language Theory. By exploring these generalized forms, this paper seeks to guide and inspire further research and development in this rapidly expanding field.

Keywords: Fuzzy set, Neutrosophic set, Hyperstructure, Hyperfuzzy set, Hyperneutrosophic set

1. Introduction

1.1. *Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets*

Set theory, a foundational pillar of mathematics, provides a robust framework for analyzing collections of elements known as "sets" [118,192,210,237,415,416]. This paper focuses on three significant extensions of set theory—Fuzzy Sets [448], Neutrosophic Sets [354], and Plithogenic Sets [367]—and explores their further generalization into Hyperfuzzy [165], HyperNeutrosophic [146], and Hyperplithogenic Sets [146]. To establish context, we begin with an overview of these foundational concepts and their approaches to handling uncertainty.

Traditional set theory has evolved over time to address the inherent complexities and ambiguities of real-world scenarios. This evolution has given rise to a variety of advanced frameworks, including Fuzzy Sets [107, 407, 448–452, 465], Vague Sets [21, 86, 95, 196, 458], Hesitant Fuzzy Sets [409, 410], Soft Sets [32, 33, 147, 256, 268, 439], Rough Sets [289–295], Neutrosophic Offsets [361, 363, 379, 380], and Neutrosophic Sets [80, 124, 279, 351, 353, 354, 356, 359, 393, 419].

These methodologies each address specific dimensions of uncertainty. For instance, *Fuzzy Sets* assign membership values ranging from 0 to 1, allowing for partial belonging and providing a flexible framework for modeling uncertainty [448]. By contrast, *Neutrosophic Sets* extend this approach by introducing three degrees—truth, indeterminacy, and falsity—offering a more comprehensive toolset for capturing ambiguity in complex systems [354, 356].

A particularly noteworthy development is the concept of *Plithogenic Sets*, which generalize these frameworks further by incorporating multi-dimensional uncertainty and contradictions. This advanced model is especially useful for addressing challenges in analyzing complex systems where traditional methodologies may fall short [3, 170, 317, 344, 367, 368, 389, 394, 403].

Significant strides have been made in the development of these uncertain set theories, and their applications have expanded into related domains. For example, uncertain graph theories such as Fuzzy Graphs and Neutrosophic Graphs have emerged as essential tools for solving a wide array of theoretical and practical problems [12–14, 17–20, 22–24, 26, 81, 137, 145–147, 149, 153–155, 327].

1.2. *Hyperstructures and Superhyperstructures*

This subsection discusses the concepts of Hyperstructures and Superhyperstructures, which represent hierarchical structures in mathematics. A *Hyperstructure* is an extension of the power set concept applied to various mathematical frameworks [387, 388]. A *Superhyperstructure*, in turn, generalizes this idea by introducing the notion of n-th power sets, thereby creating a hierarchical and iterative abstraction. Superhyperstructures build upon hyperstructural principles, allowing for deeper abstraction and complexity [387, 388].

For example, in graph theory, a *Hypergraph* is a hyperstructure, while a *Superhypergraph* extends this concept by incorporating superedges and supervertices. Hypergraphs are generalized graphs where edges, termed hyperedges, can connect more than two vertices [60, 76, 119, 171–173]. Superhypergraphs provide an even broader generalization, enabling a higher level of abstraction and flexibility in graph-theoretic studies [138–140, 140, 141, 146, 148, 150, 151, 164, 187, 188, 314, 355, 370, 371, 377, 382, 385, 387].

The discussion above focuses primarily on superhypergraphs in graph theory. However, many other concepts have been explored in the realm of superhyperstructures, including superhyperalgebras [207, 208, 350, 375, 395], superhypertopology, superhyperrings [386], superhyperrough sets [146], superhyperdecision-making [143], superhypergraph neural networks [141], superhypergroups [224], superhyperfunctions [381, 385], superhyperweighted sets [146], superhypertopologies [383, 384, 395], superhyperlanguages [135, 144], PDCA superhypercycles [138], and superhypergames [143]. These extensions represent a wide array of mathematical and applied concepts being actively studied.

1.3. *Hyperfuzzy, HyperNeutrosophic, and Hyperplithogenic Sets*

This subsection provides an explanation of Hyperfuzzy, HyperNeutrosophic, and Hyperplithogenic Sets. Fuzzy Sets [146,165,214,399], Neutrosophic Sets [146], Plithogenic Sets [146], Soft Sets [1, 136, 154, 193, 206, 304, 338, 341, 366, 380], Rough Sets [146], and Vague Sets [146] have all been extended using Hyperstructures and n-SuperHyperstructures.

For example, in the case of Fuzzy Sets, these extensions are known as Hyperfuzzy Sets [134, 165, 214–217, 249, 253, 261, 277, 399] and SuperHyperfuzzy Sets [146, 364]. Similarly, for Neutrosophic Sets, the extensions include HyperNeutrosophic Sets [146, 364] and SuperHyperNeutrosophic Sets [146, 364], while for Plithogenic Sets, they include HyperPlithogenic Sets [146] and SuperHyperPlithogenic Sets [146]. This paper examines these concepts in detail. Readers interested in the underlying principles or further details about these concepts are encouraged to consult relevant literature, such as [146], as needed.

1.4. *Our Contribution in This Paper*

This subsection presents Our Contribution in This Paper. The HyperUncertain Set and SuperHyperUncertain Set, as extensions of the Uncertain Set, are believed to have wide-ranging applications; however, research on these concepts remains in its infancy.

In this paper, we explore the applicability of HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets to various domains, including Statistics, TOPSIS, K-means Clustering, Evolutionary Theory, Topological Spaces, Decision Making, Probability, and Language Theory.

It should be noted that this paper primarily focuses on theoretical mathematical considerations. Therefore, further experimental validation and application-oriented research will be necessary for practical implementation in specific technologies and fields.

Through these studies, we sincerely hope to contribute to the advancement of future research in this domain.

2. **Preliminaries and Definitions**

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. While we aim to present the core ideas, an exhaustive exploration of all terms is beyond the scope of this work. Readers interested in further details are encouraged to consult the cited references for additional insights.

2.1. *Fundamentals of Set Theory*

This subsection offers a concise overview of the basic principles of set theory. For a more detailed treatment, we recommend established references such as [195, 210, 213].

Definition 2.1 (Set)**.** [210] A *set* is a collection of distinct and clearly defined objects, called *elements*. For any object x, it is possible to determine whether x belongs to a given set. If x is an element of a set A, this is denoted as $x \in A$. Sets are commonly represented using curly braces. For example, the set $A = \{1, 2, 3\}$ contains the elements 1, 2, and 3.

Definition 2.2 (Subset)**.** [210] A set A is said to be a *subset* of another set B, written as $A \subseteq B$, if every element of A is also an element of B. This relationship can be expressed as:

$$A \subseteq B \iff \forall x \, (x \in A \implies x \in B).$$

If $A \subseteq B$ but $A \neq B$, A is referred to as a *proper subset* of B, denoted by $A \subset B$.

Definition 2.3 (Empty Set)**.** [210] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. It is defined formally as:

$$\forall x \, (x \notin \emptyset).$$

For instance, the empty set can be represented as $\emptyset = \{\}$.

Definition 2.4 (Universal Set)**.** [210] The *universal set*, denoted by U, represents the set of all objects under consideration within a specific context. Any set A being analyzed is a subset of U. Formally:

$$A \subseteq U \quad \text{for any set } A.$$

Definition 2.5 (Operation)**.** [210] An *operation* is a function or rule that takes elements of a set S and produces another element within S. Formally, an operation $\circ$ on S is defined as:

$$\circ : S \times S \to S.$$

Examples include addition and multiplication, which are operations on the set of real numbers $\mathbb{R}$.

Definition 2.6 (Binary Operation)**.** [82] A *binary operation* on a set S is a function $* : S \times S \to S$ that combines two elements $a, b \in S$ to produce another element $a * b \in S$. For example, addition and subtraction are binary operations on $\mathbb{R}$.

2.2. *Hyperstructure and Superhyperstructure*

This subsection explores the concepts of Hyperstructure and Superhyperstructure, which serve as advanced mathematical frameworks for representing hierarchical systems. A *Hyperstructure* builds upon the notion of a powerset, providing a way to model relationships within sets. Extending this idea, a *Superhyperstructure* incorporates the n-th powerset, enabling the representation of multi-layered, hierarchical systems [135, 387, 388]. The formal definition of the n-th powerset is provided below.

Definition 2.7 (Base Set)**.** A *base set* is a fundamental set S from which more advanced structures, such as powersets and hyperstructures, are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within the specified domain}\}.$$

All elements in structures such as $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.8 (Powerset)**.** [141,325] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including both the empty set and S itself. Formally, it is defined as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.9 (n-th Powerset)**.** (cf. [141,350,387])

The n-th powerset of a set H, denoted by $P_n(H)$, is defined through an iterative process. Starting with the standard powerset, the construction proceeds as follows:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted by $P_n^*(H)$, is recursively defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set excluded.

Proposition 2.10. *(cf. [141, 350, 387])*

The n-th powerset generalizes the traditional concept of a powerset by applying the powerset operation iteratively.

Proof. The n-th powerset, $P_n(H)$, is constructed by repeatedly applying the standard powerset operation. Since $P_1(H) = P(H)$, and each subsequent step $P_{n+1}(H)$ builds upon the previous powerset, this iterative process inherently extends the original concept. Thus, the proposition follows directly from the definition. □

To provide a formal basis for the concepts of Hyperstructures and Superhyperstructures, we proceed with the following definitions and propositions.

Definition 2.11 (Classical Structure)**.** (cf. [350,387]) A *Classical Structure* is a mathematical construct defined on a non-empty set H, equipped with one or more *Classical Operations* that adhere to specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is an integer, and H^m denotes the m-fold Cartesian product of H. Examples include addition and multiplication, as found in common algebraic structures such as groups, rings, and fields.

Definition 2.12 (Hyperstructure)**.** (cf. [141,350,387]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ represents an operation defined on subsets within $\mathcal{P}(S)$.

Definition 2.13 (n-Superhyperstructure)**.** (cf. [350,387]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. It is formally represented as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation defined on elements of $\mathcal{P}_n(S)$.

Proposition 2.14. *An n-Superhyperstructure is inherently characterized by the properties of the n-th powerset.*

Proof. This follows directly from the definition of an n-Superhyperstructure, which is constructed explicitly using the n-th powerset $\mathcal{P}_n(S)$. □

Proposition 2.15. *A Hyperstructure is a special case of an n-Superhyperstructure for $n = 1$, and every n-Superhyperstructure generalizes the concept of a Hyperstructure.*

Proof. By definition, a Hyperstructure is based on the powerset $\mathcal{P}(S)$, which corresponds to the 1-st powerset $\mathcal{P}_1(S)$. When $n > 1$, the n-th powerset $\mathcal{P}_n(S)$ extends the structure, making the n-Superhyperstructure a generalization of the Hyperstructure. □

Additionally, we introduce the Dynamic n-SuperHyperStructure as an extension of the n-SuperHyperStructure framework. Although this paper does not provide a detailed discussion on the topic, it is worth noting that the concepts introduced later—such as the superhyperfuzzy set, superhyperneutrosophic set, and superhyperplithogenic set—can also be endowed with dynamic properties.

Definition 2.16 (Dynamic n-SuperHyperStructure)**.** Let S be a nonempty set, and let $\mathcal{P}_1(S) = \mathcal{P}(S)$ be the power set of S. For each integer $n \geq 2$, define:

$$\mathcal{P}_n(S) = \mathcal{P}(\mathcal{P}_{n-1}(S)),$$

the n-th power set of S.

A *Dynamic n-SuperHyperStructure* over a time domain T (which may be discrete or continuous, e.g., $T \subseteq \mathbb{R}$) is a triple:

$$\mathcal{DSH}_n = (S, \{\circ_t\}_{t\in T}, \Psi),$$

satisfying the following conditions:

(1) **Base Set and Hierarchy:** The structure is built upon the n-th powerset $\mathcal{P}_n(S)$. Each element of $\mathcal{P}_n(S)$ is a highly nested subset structure derived from S.

(2) **Time-Indexed Operation:** For each fixed $t \in T$, there is a binary operation:

$$\circ_t : \mathcal{P}_n(S) \times \mathcal{P}_n(S) \to \mathcal{P}_n(S).$$

The pair $(\mathcal{P}_n(S), \circ_t)$ is an n-SuperHyperStructure at time t. In other words, for each t, $\circ_t$ satisfies the structural axioms (if any are imposed) for n-SuperHyperStructures (e.g., closure under $\circ_t$, well-definedness of the operation, etc.).

(3) **Dynamic Evolution of Elements:** There exists an evolution function:

$$\Psi : T \times \mathcal{P}_n(S) \to \mathcal{P}_n(S),$$

which governs how elements of $\mathcal{P}_n(S)$ evolve over time. For instance, if $\{X_t\}_{t\in T}$ is a time-dependent family of elements in $\mathcal{P}_n(S)$, their evolution can be described by:

$$X_{t+\Delta t} = \Psi(t, X_t).$$

In a continuous-time setting, one might consider a limit $\Delta t \to 0$ and define a differential equation on $\mathcal{P}_n(S)$, but this requires additional structure.

(4) **Dynamic Evolution of the Operation:** The operation $\circ_t$ may also evolve with time. Formally, there is a function:

$$\Omega : T \times \mathcal{P}_n(S) \times \mathcal{P}_n(S) \to \mathcal{P}_n(S),$$

such that:

$$X \circ_t Y = \Omega(t, X, Y),$$

for all $X, Y \in \mathcal{P}_n(S)$. The choice of Ω defines how the binary operation changes as the system evolves over time.

(5) **Consistency and Well-Definedness:** For each $t \in T$, the structure $(\mathcal{P}_n(S), \circ_t)$ must be well-defined as an n-SuperHyperStructure. This means that $\circ_t$ takes two elements of $\mathcal{P}_n(S)$ and returns an element of $\mathcal{P}_n(S)$, and any additional axioms imposed on n-SuperHyperStructures (e.g., certain associative-like properties or closure properties) must hold for each fixed t.

- If Ψ is the identity function (i.e., $\Psi(t, X) = X$ for all t, X) and $\circ_t = \circ$ is time-invariant, the Dynamic n-SuperHyperStructure reduces to a static n-SuperHyperStructure.

- By appropriately choosing Ψ and Ω, one can model a wide variety of systems that exhibit hierarchical complexity and evolve over time, such as changing relational networks, adaptive classification schemes, or evolving hierarchical data structures.

2.3. *Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set*

This subsection introduces the definitions of Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set. Intuitively, these concepts extend the idea of fuzzy values into hierarchical structures, enabling a more nuanced representation of uncertainty. The formal definitions are provided below.

Definition 2.17. [448, 452] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 2.18 (Weather "Warmth")**.** Weather prediction involves forecasting atmospheric conditions, including temperature, precipitation, and wind, using scientific models, historical data, and computational methods (cf. [54, 323, 397]).

A fuzzy set τ models the degree of "warmth" in different temperature levels Y. Let:

$$Y = \{\text{Cool}, \text{Mild}, \text{Hot}\},$$

where the fuzzy set $\tau : Y \to [0, 1]$ assigns membership values to each element:

$$\tau(\text{Cool}) = 0.2, \quad \tau(\text{Mild}) = 0.6, \quad \tau(\text{Hot}) = 0.9.$$

These values represent the perceived degree of warmth for each temperature level.

Real-Life Application: A thermostat can use the fuzzy set τ to determine whether to increase cooling or heating based on the perceived warmth of the environment, rather than fixed thresholds.

Definition 2.19 (HyperFuzzy Set)**.** [72, 165, 214, 276, 399] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Example 2.20 (Customer Satisfaction with Cars)**.** Customer satisfaction measures how well a product or service meets customer expectations, influencing loyalty, retention, and overall business success (cf. [56, 100, 228]).

A hyperfuzzy set $\tilde{\mu}$ represents satisfaction levels for different car attributes X:

$$X = \{\text{Price}, \text{Fuel Efficiency}, \text{Design}\}.$$

The hyperfuzzy set $\tilde{\mu} : X \to \tilde{P}([0,1])$ assigns subsets of satisfaction degrees to each attribute:

$$\tilde{\mu}(\text{Price}) = \{0.8, 0.9\}, \quad \tilde{\mu}(\text{Fuel Efficiency}) = \{0.6, 0.7, 0.9\}, \quad \tilde{\mu}(\text{Design}) = \{0.4, 0.6\}.$$

Each subset represents possible satisfaction levels due to uncertainty or varying preferences among customers.

Real-Life Application: Car manufacturers can use this model to analyze diverse customer opinions and develop vehicles that maximize satisfaction across multiple attributes.

Definition 2.21 (n-SuperHyperFuzzy Set)**.** [**?**, 146] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 2.22 (Example: Academic Performance in Subjects)**.** Academic performance evaluates a student's success in educational tasks, measured through grades, test scores, and overall learning achievements (cf. [298, 321]).

A superhyperfuzzy set $\tilde{\mu}_n$ models learning effectiveness across subject groups X:

$$X = \{\text{Math}, \text{Science}, \text{Art}\}.$$

Level 1: The first power set $\tilde{\mathcal{P}}_1(X)$ includes subsets of X:

$$\tilde{\mathcal{P}}_1(X) = \{\{\text{Math}\}, \{\text{Science}\}, \{\text{Math, Science}\}, \dots\}.$$

For a subset $A = \{\text{Math, Science}\}$, the first-level mapping assigns:

$$\tilde{\mu}_1(A) = \{0.7, 0.8\}.$$

Level 2: The second power set $\tilde{\mathcal{P}}_2(X)$ contains subsets of $\tilde{\mathcal{P}}_1(X)$. For example:

$$\tilde{\mathcal{P}}_2(X) = \{\{\{\text{Math}\}, \{\text{Science}\}\}, \{\{\text{Math, Science}\}\}, \dots\}.$$

For $B = \{\{\text{Math}\}, \{\text{Science}\}\}$, the second-level mapping assigns:

$$\tilde{\mu}_2(B) = \{0.6, 0.9\}.$$

Real-Life Application: Educational institutions can model and optimize the learning effectiveness of combined subject groups, designing curricula that cater to hierarchical dependencies between disciplines.

Theorem 2.23. *An n-superhyperfuzzy set generalizes both fuzzy sets and hyperfuzzy sets.*

Proof. This is evident. Refer to [146] if necessary. □

As demonstrated above, Fuzzy Sets can be extended to Hyperfuzzy Sets and SuperHyperfuzzy Sets, allowing related concepts to be similarly generalized. For instance, Bipolar Fuzzy Sets [27, 67, 68, 318, 459–461] can be extended to Bipolar Hyperfuzzy Sets and Bipolar SuperHyperfuzzy Sets; Tripolar Fuzzy Sets [308–311] to Tripolar Hyperfuzzy Sets and Tripolar SuperHyperfuzzy Sets; Neuro-Fuzzy Sets [11, 96, 128, 185, 197, 227, 267] to Neuro-Hyperfuzzy Sets and Neuro-SuperHyperfuzzy Sets; Hesitant Fuzzy SetsSets [324, 334, 409, 410, 437] to Hesitant Hyperfuzzy Sets and Hesitant SuperHyperfuzzy Sets; Picture Fuzzy Sets [103,104,159,251,335, 347,398,422,426] to Picture Hyperfuzzy Sets and Picture SuperHyperfuzzy Sets; Pythagorean Fuzzy Sets [296, 427, 462] to Pythagorean Hyperfuzzy Sets and Pythagorean SuperHyperfuzzy Sets; Spherical Fuzzy Sets [34, 41, 130, 180, 181, 254, 262, 418] to Spherical Hyperfuzzy Sets and Spherical SuperHyperfuzzy Sets; and Nonstationary Fuzzy Sets [37, 161, 200, 201, 322, 443]to Nonstationary Hyperfuzzy Sets and Nonstationary SuperHyperfuzzy Sets. These extensions provide a framework for addressing increasingly complex and nuanced forms of uncertainty.

2.4. *Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets*

Building on the concept of Fuzzy Sets, this subsection explores Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets. Neutrosophic Sets extend Fuzzy Sets by incorporating the notion of indeterminacy, which captures situations that are neither true nor false [354]. This generalization provides a more robust framework for modeling real-world scenarios characterized by higher levels of uncertainty and complexity than Fuzzy Sets can address. Consequently, these concepts have been the focus of extensive research in numerous studies [156, 157, 225, 352, 356, 357, 373, 390–392, 396]. Relevant definitions are provided below.

Definition 2.24 (Neutrosophic Set)**.** [354] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.25 (A decision-making scenario in everyday life)**.** Decision-making is the process of evaluating options and selecting the best course of action to achieve specific goals or solve problems (cf. [28, 92, 297]).

Consider a decision-making scenario in everyday life: determining whether a given fruit (say, a banana) is "ripe" enough to eat. Let X be the set of all bananas in a grocery store. Define a Neutrosophic Set A representing the subset of bananas that are considered "ripe."

For each banana $x \in X$:

$T_A(x) \in [0,1]$ measures the degree of certainty that the banana is ripe (e.g., based on color and smell).

$I_A(x) \in [0,1]$ measures the indeterminacy or uncertainty about its ripeness (e.g., poor lighting, no direct smell test

$F_A(x) \in [0,1]$ measures the degree of certainty that the banana is not ripe (e.g., it is still green and too hard).

A possible assignment might be:

$$T_A(x) = 0.7, \quad I_A(x) = 0.2, \quad F_A(x) = 0.1,$$

indicating a reasonably high belief that the banana is ripe, some uncertainty due to ambiguous appearance, and a small possibility that it is not ripe.

Definition 2.26 (HyperNeutrosophic Set)**.** [146, 364] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Example 2.27. Now consider extending the scenario. Suppose there are multiple experts or tests evaluating the same banana's ripeness. One expert relies on color, another on smell, another on texture. Each produces a triplet (T, I, F) of evaluations. For each banana x, instead of assigning a single (T, I, F), we now have a *set* of such triplets, reflecting various perspectives.

For example:

$$\tilde{\mu}(x) = \{(0.8, 0.1, 0.1), (0.6, 0.3, 0.1), (0.75, 0.2, 0.05)\},$$

where each element in $\tilde{\mu}(x)$ might come from a different test:

- First triplet (0.8, 0.1, 0.1): Based on color analysis.
- Second triplet (0.6, 0.3, 0.1): Based on a smell test.
- Third triplet (0.75, 0.2, 0.05): Based on texture and firmness test.

Combining these different evaluations into one set for each banana gives a HyperNeutrosophic Set. Thus, each banana's "membership" in the concept of "ripe" is represented not just by one triple, but by a collection of triples capturing multiple dimensions of assessment.

Definition 2.28 (n-SuperHyperNeutrosophic Set)**.** [146, 364] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 2.29. Consider a supply chain scenario (cf. [263, 400]) involving multiple stages:

(1) Farms where bananas are grown.
(2) Distribution centers where bananas from different farms are aggregated.
(3) Retailers who receive these bananas and sell them to consumers.

We can model the degrees of banana ripeness at different layers of this supply chain using Neutrosophic-type sets. If we let $n = 0$ correspond to a Neutrosophic Set, then at the farm stage (where $n = 0$), each banana is associated with a single (T, I, F)-triple indicating its ripeness:

$$\tilde{A}_0 : \tilde{\mathcal{P}}_0(X) \to \tilde{\mathcal{P}}_0([0,1]^3),$$

where $\tilde{\mathcal{P}}_0(X)$ is just X itself, and $\tilde{\mathcal{P}}_0([0,1]^3)$ represents assigning one (T, I, F)-triple per element.

At the distribution center stage (where $n = 1$), you aggregate the farm-level (Neutrosophic) evaluations into a HyperNeutrosophic Set. This means that for each banana, instead of having

just one (T, I, F)-triple, you now have a *set* of such triples (e.g., from different assessment criteria like color, smell, and texture):

$$\tilde{A}_1 : \tilde{\mathcal{P}}_1(X) \to \tilde{\mathcal{P}}_1([0,1]^3),$$

where $\tilde{\mathcal{P}}_1(X)$ indicates we are considering sets of elements with potentially multiple (T, I, F)-triplets assigned to each.

At the retail stage (where $n = 2$ or higher), you can form an n-SuperHyperNeutrosophic Set by integrating data from various distribution centers. This yields nested structures of sets of sets of (T, I, F)-triplets, capturing multiple layers of uncertainty and information aggregation:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where $\tilde{\mathcal{P}}_n(X)$ represents n-fold nested sets, and each nesting corresponds to an additional layer of aggregation.

For example, at a high n (several stages beyond just farms and distribution centers), you might have:

$$\tilde{A}_n(A) = \{\{\{\dots(T_1, I_1, F_1), (T_2, I_2, F_2), \dots\}\dots\}\},$$

with multiple nested collections of (T, I, F)-triplets acquired from multiple levels of the supply chain. Each nesting step adds more complexity, reflecting a combination of many different assessments and conditions contributing to the final understanding of banana ripeness across the entire supply chain.

Theorem 2.30. *An n-superhyperneutrosophic set generalizes both neutrosophic sets and hyperneutrosophic sets.*

Proof. This is evident. Refer to [146] if necessary. □

Theorem 2.31. *An n-superhyperneutrosophic set generalizes n-superhyperfuzzy sets.*

Proof. This is evident. Refer to [146] if necessary. □

As demonstrated above, Neutrosophic Sets can be extended to Hyperneutrosophic Sets and SuperHyperneutrosophic Sets, enabling the generalization of related concepts. For example, Bipolar Neutrosophic Sets [112, 113, 413] can be extended to Bipolar Hyperneutrosophic Sets and Bipolar n-SuperHyperneutrosophic Sets. Similarly, Interval-Valued Neutrosophic Sets [332, 442, 447, 454, 456] can be generalized to Interval-Valued HyperNeutrosophic Sets and Interval-Valued n-SuperHyperneutrosophic Sets. Furthermore, Neutrosophic Soft Sets [223,

255, 333] can be extended to Hyperneutrosophic Soft Sets and n-SuperHyperneutrosophic Soft Sets.

Additionally, numerous types of Neutrosophic Sets are known, such as Hesitant Neutrosophic Sets [331, 463], Spherical Neutrosophic Sets [64, 169, 234, 326], and Intuitionistic Neutrosophic Sets [62, 63, 202]. These sets can similarly be extended, offering a broader framework for analyzing and applying Neutrosophic concepts across various fields.

2.5. *HyperPlithogenic Set*

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [367, 368]. The definition of the Plithogenic Set is provided below.

Definition 2.32. [367, 368] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Example 2.33. (cf. [142, 153]) The following examples of Plithogenic Sets are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Set*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Set*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Set*.

Definition 2.34 (HyperPlithogenic Set)**.** [146] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : (\bigcup_{i=1}^n Pv_i) \times (\bigcup_{i=1}^n Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 2.35 (n-SuperHyperPlithogenic Set)**.** [146] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  (1) Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
  (2) Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Theorem 2.36. *An n-superhyperplithogenic set generalizes both plithogenic sets and hyperplithogenic sets.*

Proof. This is evident. Refer to [146] if necessary. □

Theorem 2.37. *An n-superhyperplithogenic set generalizes both n-superhyperfuzzy sets and n-superhyperneutrosophic sets.*

Proof. This is evident. Refer to [146] if necessary. □

2.6. *Plithogenic Sets: A Unified Framework for Handling Various Parameters*

Plithogenic Sets provide a unified framework for generalizing a wide range of concepts, as described below. This capability extends naturally to HyperPlithogenic Sets and n-SuperHyperPlithogenic Sets, maintaining the same principles of generalization.

Example 2.38. (cf. [142,153]) The following table provides concrete examples of Plithogenic Sets and their transformations, highlighting the number of parameters (s) and their associated features. These examples also illustrate the diversity and complexity of Plithogenic Sets.

It is important to emphasize that while the number of parameters is relevant for classification, their names and meanings must also be considered, particularly when analyzing applications. For example, the Quadripartitioned Neutrosophic Set is known to be more practical and applicable compared to Turiyam Neutrosophic Sets or Ambiguous Sets [351]. Additionally, it should be noted that other extended sets, such as the Multi-fuzzy set [29,35,339,340,440] and the Refined Neutrosophic Set [111,220,275,313,412,414], also exist. These sets include multiple parameters representing the same type of uncertainty, further enriching their descriptive capabilities.

s	t	*Type of Plithogenic Set*
1	1	Fuzzy Set [448]
2	1	Intuitionistic Fuzzy Set [46–53] Vague Set [86,95,196] Paraconsistent Set [87,131,247,424,425]
3	1	Neutrosophic Set [353,354] Hesitant Fuzzy Set [409,410] Three-way Fuzzy Set [199,420] Spherical Fuzzy Set [24,179,181]
4	1	Quadripartitioned Neutrosophic Set [71,90] Double-Valued Neutrosophic Set [120,219,230,252,428] Ambiguous Set [345,346,351] Turiyam Neutrosophic Set [158,343]
5	1	Pentapartitioned Neutrosophic Set [106]
6	1	Hexapartitioned Neutrosophic Set [288]
7	1	Heptapartitioned Neutrosophic Set [78,271]
8	1	Octapartitioned Neutrosophic Set
9	1	Nonapartitioned Neutrosophic Set

TABLE 1. Examples of Plithogenic Sets based on parameters s and t.

3. Result: Application of HyperNeutrosophic Sets to Various Sciences

This paper explores the application of HyperNeutrosophic concepts across various scientific fields. It is worth noting that if a HyperNeutrosophic Set can be applied to a particular domain, it is natural to consider that Hyperfuzzy Sets and Hyperplithogenic Sets could also find application in similar contexts. Furthermore, for SuperHyperNeutrosophic Sets, it is equally plausible to examine the applicability of SuperHyperfuzzy Sets and SuperHyperplithogenic Sets within the same or related domains.

3.1. *HyperNeutrosophic Probability*

Probability quantifies the likelihood of an event occurring, represented as a value between 0 (impossible) and 1 (certain) [110, 264, 284, 285, 299, 306]. Related concepts, such as Hyper-Probability and n-SuperHyperProbability, have also been explored [85, 144].

Furthermore, extended frameworks like Fuzzy Probability [84, 245, 300, 330, 457, 464], Neutrosophic Probability [358, 362], and Plithogenic Probability [365] are well-established. These concepts will be analyzed in the following sections.

Definition 3.1 (Neutrosophic Probability)**.** [358, 362] Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω (events). A *Neutrosophic Probability (NP)* is a function:

$$NP : \mathcal{F} \to [0, 1]^3,$$

where for each event $A \in \mathcal{F}$, $NP(A) = (T(A), I(A), F(A))$ satisfies:

$$0 \leq T(A), I(A), F(A) \leq 1 \quad \text{and} \quad 0 \leq T(A) + I(A) + F(A) \leq 3.$$

Here:

- $T(A)$: Truth degree of A (likelihood of A being true),
- $I(A)$: Indeterminacy degree of A (uncertainty about A),
- $F(A)$: Falsity degree of A (likelihood of A being false).

Example 3.2 (Applications of Neutrosophic Probability)**.** This section provides two precise examples of Neutrosophic Probability and their applications in real-life contexts.

1. Weather Forecasting: Consider $\Omega = \{\text{rain, sunny, cloudy}\}$, and let the event $A = \{\text{rain}\}$ represent the occurrence of rain. A Neutrosophic Probability function assigns:

$$NP(A) = (0.75, 0.15, 0.10),$$

where:

- $T(A) = 0.75$: The degree of truth, representing the probability of rain based on meteorological predictions and atmospheric data.

- $I(A) = 0.15$: The indeterminacy, accounting for uncertainty due to limited data or conflicting weather models.
- $F(A) = 0.10$: The degree of falsity, indicating the probability that rain will not occur based on historical weather patterns.

This enables a nuanced understanding of the likelihood of rain, integrating truth, uncertainty, and contradiction.

2. Medical Diagnosis: In a diagnostic scenario, consider $\Omega = \{\text{healthy, sick}\}$ with $A = \{\text{sick}\}$ representing a positive diagnosis for an illness. A Neutrosophic Probability function assigns:

$$NP(A) = (0.80, 0.10, 0.10),$$

where:

- $T(A) = 0.80$: The probability of being sick based on test results.
- $I(A) = 0.10$: Indeterminacy arising from inconclusive or incomplete diagnostic information.
- $F(A) = 0.10$: The probability of being healthy despite the test results, indicating potential errors or anomalies in testing.

This provides a comprehensive representation of diagnostic uncertainty, enhancing clinical decision-making.

Definition 3.3 (HyperNeutrosophic Probability)**.** Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω. A *HyperNeutrosophic Probability (HNP)* is a function:

$$HNP : \mathcal{F} \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the power set of the neutrosophic unit cube $[0,1]^3$. For each event $A \in \mathcal{F}$, $HNP(A)$ is defined as:

$$HNP(A) = \{(T_k(A), I_k(A), F_k(A)) \mid k \in \mathcal{K}_A\},$$

where:

- $T_k(A), I_k(A), F_k(A) \in [0,1]$ are the k-th truth, indeterminacy, and falsity degrees of A,
- $\mathcal{K}_A$ is the index set of evaluations for A,
- Each triple satisfies:

$$0 \leq T_k(A) + I_k(A) + F_k(A) \leq 3.$$

Example 3.4 (Applications of HyperNeutrosophic Probability)**.** This section elaborates on two concrete examples of HyperNeutrosophic Probability, illustrating its utility in real-world scenarios.

1. Financial Risk Assessment (cf. [236]): In assessing credit default risks, let $\Omega = \{\text{default, no default}\}$ with $A = \{\text{default}\}$ representing the event of a credit default. A HyperNeutrosophic Probability function assigns:

$$HNP(A) = \{(0.85, 0.10, 0.05), (0.75, 0.15, 0.10), (0.80, 0.12, 0.08)\},$$

where:

- Each triple $(T_k(A), I_k(A), F_k(A))$ corresponds to evaluations from different financial models or analysts.
- $T_k(A)$: Reflects the degree of truth (likelihood of default based on financial metrics).
- $I_k(A)$: Captures uncertainty due to varying assumptions or incomplete data.
- $F_k(A)$: Represents the likelihood of no default, considering optimistic scenarios or mitigating factors.

This enables risk analysis by consolidating diverse expert perspectives.

2. Environmental Risk Analysis(cf. [178]): In evaluating the impact of industrial activities, let $\Omega = \{\text{low impact, medium impact, high impact}\}$ with $A = \{\text{high impact}\}$ representing a severe environmental impact. A HyperNeutrosophic Probability function assigns:

$$HNP(A) = \{(0.65, 0.20, 0.15), (0.70, 0.15, 0.15), (0.60, 0.25, 0.15)\},$$

where:

- Each triple corresponds to evaluations from different environmental studies or simulation models.
- $T_k(A)$: Degree of truth based on environmental assessments or emissions data.
- $I_k(A)$: Indeterminacy arising from uncertain or conflicting data.
- $F_k(A)$: Degree of falsity representing the likelihood of minimal or no impact.

This facilitates a robust decision-making process by integrating multiple environmental perspectives and addressing uncertainties.

Definition 3.5 (n-SuperHyperNeutrosophic Probability)**.** Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω. An *n-SuperHyperNeutrosophic Probability (n-SHNP)* is a function:

$$HNP^{(n)} : \mathcal{F} \to \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ is the n-th nested power set of $[0,1]^3$. For each event $A \in \mathcal{F}$, $HNP^{(n)}(A)$ is defined recursively as:

$$HNP^{(n)}(A) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(HNP^{(n-1)}(A)), & \text{if } n \geq 1. \end{cases}$$

This structure captures higher-order uncertainties and allows for multi-layered probability evaluations.

Theorem 3.6 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Probability possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.7 (Reduction to Neutrosophic Probability)**.** *For $n = 0$, n-SuperHyperNeutrosophic Probability (n-SHNP) reduces to standard Neutrosophic Probability (NP).*

Proof. By definition, when $n = 0$, we have:

$$HNP^{(0)}(A) = [0, 1]^3,$$

where $[0, 1]^3$ represents a single neutrosophic triple $(T(A), I(A), F(A))$ with:

$$0 \leq T(A), I(A), F(A) \leq 1 \quad \text{and} \quad 0 \leq T(A) + I(A) + F(A) \leq 3.$$

This is precisely the definition of a Neutrosophic Probability for the event A. Therefore, 0-SHNP is equivalent to NP. □

Theorem 3.8 (Hierarchical Complexity)**.** *For every $n \geq 1$, n-SHNP is strictly more complex, in terms of set structure, than $(n-1)$-SHNP.*

Proof. Consider the recursive definition:

$$HNP^{(n)}(A) = \mathcal{P}(HNP^{(n-1)}(A)).$$

When moving from $(n-1)$ to n, we apply the power set operation $\mathcal{P}$, which generates the set of all subsets of $HNP^{(n-1)}(A)$. The power set operation always produces a strictly larger and more complex structure for non-empty sets. Hence, $HNP^{(n)}(A)$ introduces an additional layer of combinatorial and hierarchical complexity compared to $HNP^{(n-1)}(A)$, making n-SHNP strictly more complex than $(n-1)$-SHNP. □

Theorem 3.9 (Consistency of Truth, Indeterminacy, and Falsity)**.** *For any event $A \in \mathcal{F}$, the total measure of truth, indeterminacy, and falsity at each level n satisfies:*

$$0 \leq \sum_{(T,I,F) \in HNP^{(n)}(A)} \big(T + I + F\big) \leq 3 \cdot |HNP^{(n)}(A)|.$$

Proof. By definition, each triple $(T_k, I_k, F_k) \in HNP^{(n)}(A)$ satisfies:

$$0 \leq T_k + I_k + F_k \leq 3.$$

For the n-th level, $HNP^{(n)}(A)$ consists of subsets of $(n-1)$-SHNP structures due to the power set operation. The total measure over all elements in $HNP^{(n)}(A)$ is given by:

$$\sum_{(T,I,F)\in HNP^{(n)}(A)} \big(T+I+F\big),$$

which must be bounded above by $3 \cdot |HNP^{(n)}(A)|$ due to the constraints on each triple. The lower bound 0 is achieved if all measures are zero. This consistency holds at all levels n. □

Theorem 3.10 (Projection Property)**.** *For any $n \geq 1$, there exists a projection function:*

$$\pi_n : HNP^{(n)}(A) \to HNP^{(n-1)}(A),$$

which maps an n-SHNP structure to its corresponding $(n-1)$-SHNP structure without violating the probability constraints.

Proof. Define π_n as the inverse of the power set operation:

$$\pi_n(S) = \{x \in S \mid x \in HNP^{(n-1)}(A)\}.$$

By construction, $HNP^{(n)}(A) = \mathcal{P}(HNP^{(n-1)}(A))$, so every element of $HNP^{(n)}(A)$ is a subset of $HNP^{(n-1)}(A)$. Applying π_n preserves the original structure of $HNP^{(n-1)}(A)$. The constraints on truth, indeterminacy, and falsity are inherited directly from the $(n-1)$-level, ensuring that π_n maintains consistency with the probability framework. □

Theorem 3.11 (Continuity under Nested Refinement)**.** *If the probability values (T, I, F) are adjusted within the bounds of the constraints at level n, the resulting n-SHNP structure varies continuously with respect to those adjustments.*

Proof. Each level n of the n-SHNP structure is defined recursively via the power set operation. Small changes in the neutrosophic values (T, I, F) at any level propagate through the hierarchy as adjustments to the subsets in subsequent levels. Since the power set operation involves discrete combinations of subsets, the adjustments in the probability values lead to corresponding changes in the elements of the higher-level power sets without discontinuities. Thus, the structure of $HNP^{(n)}(A)$ varies continuously under bounded perturbations of the probability values at any level. □

Similarly, HyperFuzzy Probability, HyperPlithogenic Probability, SuperHyperFuzzy Probability, and SuperHyperPlithogenic Probability can also be defined.

3.2. *HyperNeutrosophic Statistics*

This subsection introduces HyperNeutrosophic Statistics. Statistics is the discipline concerned with the collection, organization, analysis, and interpretation of data, aiding in making informed decisions and predictions [74, 75, 429]. Statistics is deeply connected to probability theory and encompasses important concepts such as random variables [284, 299], expectations [61, 243], and distribution functions [121, 246].

Extensions of classical statistics, such as Fuzzy Statistics [83, 115, 221, 239, 278, 307, 408, 423], Neutrosophic Statistics [42–45, 93, 360], and Plithogenic Statistics [369, 374, 378], have been developed to address various forms of uncertainty. This paper further extends these concepts to HyperNeutrosophic Statistics and SuperHyperNeutrosophic Statistics. Relevant definitions are provided below.

Definition 3.12 (Neutrosophic Statistics)**.** [93, 360] Neutrosophic Statistics (NS) is a generalization of classical statistics and interval statistics, developed to handle data characterized by indeterminacy, vagueness, and conflicting information. It relies on Neutrosophic Logic, Probability, and Set Theory to provide a robust framework for analyzing uncertain datasets. Components of Neutrosophic Statistics.

(1) *Neutrosophic Probability Distribution (NPD):* For a random event x, its occurrence is described by a triple:

$$NPD(x) = \big(T(x), I(x), F(x)\big),$$

where:

- $T(x)$: The degree of truth (probability that x occurs).
- $I(x)$: The degree of indeterminacy (uncertainty whether x occurs or not).
- $F(x)$: The degree of falsity (probability that x does not occur).

These values satisfy the condition:

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

(2) *Neutrosophic Random Variable (NRV) [79, 174, 175]:* A random variable X in Neutrosophic Statistics maps outcomes to their corresponding neutrosophic probabilities:

$$X : \Omega \to \{(T, I, F) \in [0, 1]^3 \mid 0 \leq T + I + F \leq 3\}.$$

Unlike classical random variables, NRVs include indeterminate components, enabling more flexible representations of uncertainty.

(3) *Neutrosophic Expectation:* The expectation $E[X]$ of a neutrosophic random variable X is defined as:

$$E[X] = \big(E_T[X], E_I[X], E_F[X]\big),$$

where $E_T[X]$, $E_I[X]$, and $E_F[X]$ represent the expectations of truth, indeterminacy, and falsity, respectively.

(4) *Neutrosophic Variance:* The variance $\mathrm{Var}[X]$ is computed component-wise:

$$\mathrm{Var}[X] = \big(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big),$$

providing a detailed analysis of data uncertainty.

(5) *Neutrosophic Distribution Functions [2]:*

- The *Neutrosophic Probability Density Function (N-PDF)* is defined as:

$$f_X(x) = \big(f_T(x), f_I(x), f_F(x)\big),$$

where $f_T(x)$, $f_I(x)$, and $f_F(x)$ represent the densities for truth, indeterminacy, and falsity.

- The *Cumulative Neutrosophic Distribution Function (N-CDF)*:

$$F_X(x) = \big(F_T(x), F_I(x), F_F(x)\big),$$

accumulates probabilities across truth, indeterminacy, and falsity domains.

Example 3.13 (Neutrosophic Statistics: Disease Diagnosis)**.** Consider a medical diagnosis scenario (cf. [411]) where the sample space is $\Omega = \{\text{disease present}, \text{disease absent}\}$. Let $A = \{\text{disease present}\}$ represent the event of interest.

A Neutrosophic Probability Distribution (NPD) for A could be:

$$NPD(A) = (T(A), I(A), F(A)) = (0.7, 0.2, 0.1),$$

where:

- $T(A) = 0.7$: Probability that the patient truly has the disease.
- $I(A) = 0.2$: Uncertainty due to inconclusive test results.
- $F(A) = 0.1$: Probability that the patient does not have the disease.

Let a Neutrosophic Random Variable X represent the severity level of the disease on a scale from 0 to 10. For instance:

$$X(\text{disease present}) = (0.7, 0.2, 0.1), \quad X(\text{disease absent}) = (0.1, 0.3, 0.6).$$

Assuming severity level 8 for "disease present" and 0 for "disease absent," the Neutrosophic Expectation is:

$$E_T[X] = 0.7{\times}8{+}0.1{\times}0 = 5.6, \quad E_I[X] = 0.2{\times}8{+}0.3{\times}0 = 1.6, \quad E_F[X] = 0.1{\times}8{+}0.6{\times}0 = 0.8.$$

Thus, the expectation is:

$$E[X] = (5.6, 1.6, 0.8),$$

providing a comprehensive analysis of the severity with truth, indeterminacy, and falsity components.

Definition 3.14 (HyperNeutrosophic Statistics)**.** HyperNeutrosophic Statistics (HNS) is a framework that generalizes Neutrosophic Statistics by incorporating sets of Neutrosophic Probability Distributions (NPDs) to represent multiple sources of uncertainty or criteria simultaneously.

(1) *HyperNeutrosophic Probability Distribution (HNPD):* Let $x \in \Omega$ represent a random event. A HyperNeutrosophic Probability Distribution (HNPD) for x is defined as:

$$HNPD(x) = \{\big(T_i(x), I_i(x), F_i(x)\big) \mid i \in \mathcal{I}\},$$

where:

- $\mathcal{I}$ is an index set representing distinct sources, tests, or criteria.
- Each triple $\big(T_i(x), I_i(x), F_i(x)\big)$ satisfies:

$$0 \leq T_i(x), I_i(x), F_i(x) \leq 1 \quad \text{and} \quad 0 \leq T_i(x) + I_i(x) + F_i(x) \leq 3.$$

(2) *HyperNeutrosophic Random Variable (HNRV):* A HyperNeutrosophic Random Variable X maps outcomes to their corresponding HNPDs:

$$X : \Omega \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the power set of $[0,1]^3$, representing all possible sets of neutrosophic probabilities.

(3) *HyperNeutrosophic Expectation and Variance:* Let X be an HNRV. The expectation $E[X]$ and variance $\mathrm{Var}[X]$ are defined component-wise over all $i \in \mathcal{I}$:

$$E[X] = \big\{E_T[X], E_I[X], E_F[X]\big\}, \quad \text{where} \quad E_T[X] = \sum_{i \in \mathcal{I}} T_i(x),$$

and similarly for $E_I[X]$ and $E_F[X]$. The variance is given by:

$$\mathrm{Var}[X] = \big\{\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big\}.$$

Example 3.15 (HyperNeutrosophic Statistics: Multi-Test Disease Diagnosis)**.** Now consider the same scenario, but with three independent diagnostic tests $\mathcal{I} = \{1, 2, 3\}$.

A HyperNeutrosophic Probability Distribution (HNPD) for $A = \{\text{disease present}\}$ is:

$$HNPD(A) = \{(0.65, 0.25, 0.10), (0.75, 0.15, 0.10), (0.70, 0.20, 0.10)\},$$

where:

- Each triple $(T_i(A), I_i(A), F_i(A))$ corresponds to a test:
  - Test 1: $(0.65, 0.25, 0.10)$,
  - Test 2: $(0.75, 0.15, 0.10)$,
  - Test 3: $(0.70, 0.20, 0.10)$.

A HyperNeutrosophic Random Variable X maps outcomes to sets of triples. For "disease present," the set of probabilities is:

$$X(\text{disease present}) = HNPD(A).$$

Using severity level 8 for "disease present" and 0 for "disease absent," the aggregated expectations are:

$$E_T[X] = \frac{8 \times (0.65 + 0.75 + 0.70)}{3} = 5.6, \quad E_I[X] = \frac{8 \times (0.25 + 0.15 + 0.20)}{3} = 1.6, \quad E_F[X] = \frac{8 \times (0.10 + 0.10 + 0.}{3}$$

Thus, the HyperNeutrosophic Expectation is:

$$E[X] = (5.6, 1.6, 0.8),$$

capturing uncertainty across multiple diagnostic tests and providing a robust analysis.

Definition 3.16 (n-SuperHyperNeutrosophic Statistics)**.** n-SuperHyperNeutrosophic Statistics (n-SHNS) is a hierarchical generalization of HyperNeutrosophic Statistics (HNS). It introduces n-fold nested hyperstructures derived from Neutrosophic Probability Distributions (NPDs), allowing for multiple, recursively defined levels of uncertainty.

(1) *n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD):* Let $x \in \Omega$ be a random event. A HyperNeutrosophic Probability Distribution (HNPD) is defined as a set of Neutrosophic Probability Distributions:

$$HNPD(x) = \{(T_i(x), I_i(x), F_i(x)) \mid i \in \mathcal{I}\},$$

where each triple $(T_i(x), I_i(x), F_i(x))$ corresponds to a neutrosophic characterization of x under the i-th source of uncertainty, with:

$$0 \leq T_i(x), I_i(x), F_i(x) \leq 1 \quad \text{and} \quad 0 \leq T_i(x) + I_i(x) + F_i(x) \leq 3.$$

An n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD) is defined by applying the power set operation n times to the HNPD structure:

$$n\text{-SHNPD}(x) = \mathcal{P}^n\big(HNPD(x)\big),$$

where:

$$\mathcal{P}^1(HNPD(x)) = \mathcal{P}(HNPD(x)), \quad \mathcal{P}^2(HNPD(x)) = \mathcal{P}(\mathcal{P}(HNPD(x))), \quad \text{and so forth.}$$

Thus, for $n \geq 1$, n-SHNPD consists of n-fold nested sets of sets of neutrosophic triples, representing progressively more complex and layered uncertainty structures.

(2) *n-SuperHyperNeutrosophic Random Variable (n-SHNRV):* An n-SuperHyperNeutrosophic Random Variable X maps each outcome $\omega \in \Omega$ to an n-SHNPD:

$$X : \Omega \to n\text{-SHNPD}(x).$$

(3) *n-SuperHyperNeutrosophic Expectation and Variance:* The expectation $E[X]$ of an n-SHNRV is defined by recursively applying the expectation operator within the nested power sets, aggregating truth, indeterminacy, and falsity components at each level. Similarly, the variance $\mathrm{Var}[X]$ is defined by recursively computing and aggregating variances across all nested layers of uncertainty:

$$E[X] = \mathcal{P}^n\big(E_T[X], E_I[X], E_F[X]\big), \quad \mathrm{Var}[X] = \mathcal{P}^n\big(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big).$$

(4) *Recursive Structure:* For $k > 1$, a k-SuperHyperNeutrosophic Probability Distribution is defined as:

$$k\text{-SHNPD}(x) = \mathcal{P}((k-1)\text{-SHNPD}(x)).$$

By construction:

- For $n = 0$, n-SHNS reduces to Neutrosophic Statistics.
- For $n = 1$, it reduces to HyperNeutrosophic Statistics.
- For $n > 1$, it provides increasingly complex hierarchical structures to model uncertainty.

Theorem 3.17 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Statistics possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.18 (Reduction to Neutrosophic Statistics)**.** *For $n = 0$, n-SuperHyperNeutrosophic Statistics (SHNS) reduces to standard Neutrosophic Statistics.*

Proof. By definition, the n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD) is constructed by applying n-fold nested power set operations to a HyperNeutrosophic Probability Distribution (HNPD). For $n = 0$, no power set operations are applied. This leaves us with the original Neutrosophic Probability Distribution (NPD), thus recovering the standard Neutrosophic Statistics framework as defined in [93, 360]. □

Theorem 3.19 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SHNPD is strictly more complex (in terms of hierarchical nesting) than the $(n-1)$-SHNPD.*

Proof. Consider $(n-1)$-SHNPD, defined as $(n-1)$-fold nested power sets of an HNPD. To obtain n-SHNPD, we apply one more power set operation:

$$n\text{-SHNPD}(x) = \mathcal{P}((n-1)\text{-SHNPD}(x)).$$

The power set operation $\mathcal{P}$ applied to any non-empty set S produces a strictly larger and more complex family of subsets. Thus, n-SHNPD is a strictly more nested and richer structure than

$(n-1)$-SHNPD. This increased complexity follows from the exponential growth in the number of subsets at each level of the hierarchy. □

Theorem 3.20 (Consistency of Expectation and Variance under Projection)**.** *Let X be an n-SuperHyperNeutrosophic Random Variable (n-SHNRV). The expectation $E[X]$ and variance $Var[X]$ defined at the n-th level are consistent with those at the $(n-1)$-th level under suitable projection.*

Proof. By definition, the expectation and variance at the n-th level, $E[X]$ and $\mathrm{Var}[X]$, are formed by applying the expectation and variance operations within nested power sets of neutrosophic triples:

$$E[X] = \mathcal{P}^n(E_T[X], E_I[X], E_F[X]), \quad \mathrm{Var}[X] = \mathcal{P}^n(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

If we consider a suitable "flattening" or projection operation π that reduces $\mathcal{P}^n$ to $\mathcal{P}^{n-1}$ by selecting representative elements or subsets at each level, the resulting structures satisfy:

$$\pi(E[X]) = \mathcal{P}^{n-1}(E_T[X], E_I[X], E_F[X]), \quad \pi(\mathrm{Var}[X]) = \mathcal{P}^{n-1}(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

These projected forms coincide with the expectations and variances defined at the $(n-1)$-th level. Hence, the operations are consistent across levels, maintaining hierarchical coherence of the statistical parameters. □

Theorem 3.21 (Non-emptiness of Expectation and Variance)**.** *For any n-SHNRV X, the sets $E[X]$ and $Var[X]$ are always well-defined and non-empty.*

Proof. Each level of the n-SHNRV is built upon non-empty sets of neutrosophic triples, as $HNPD(x)$ and its subsequent power sets $\mathcal{P}^k(HNPD(x))$ are by definition non-empty. The expectation and variance are computed by aggregating these neutrosophic values:

$$E[X] = \mathcal{P}^n(E_T[X], E_I[X], E_F[X]), \quad \mathrm{Var}[X] = \mathcal{P}^n(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

Since $(E_T[X], E_I[X], E_F[X])$ and $(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X])$ are derived from non-empty neutrosophic distributions, and the power set operation $\mathcal{P}$ applied to a non-empty set never results in an empty set, it follows that both $E[X]$ and $\mathrm{Var}[X]$ are well-defined and non-empty at every level n. □

Similarly, HyperFuzzy Statistics, HyperPlithogenic Statistics, SuperHyperFuzzy Statistics, and SuperHyperPlithogenic Statistics can also be defined.

3.3. *Neutrosophic Decision Making Theory*

Social Choice Theory is a branch of economics and political science that examines collective decision-making processes, where individual preferences are aggregated to determine a collective choice or societal welfare [39, 88, 269]. Decision-making, in this context, refers to the process of evaluating alternatives and selecting the most suitable option to achieve a desired objective [122, 123, 166, 259]. Extensions of traditional decision-making frameworks, such as Hyperdecision-Making and SuperHyperdecision-Making, have been developed to tackle complex systems involving hierarchical and multi-layered uncertainties [135]. Furthermore, decision-making has been extensively studied from perspectives such as Fuzzy and Neutrosophic frameworks, highlighting its importance across diverse domains [15, 16, 24, 25, 40, 270, 402, 445, 453].

In this paper, we extend Neutrosophic Decision Making to handle more intricate uncertainties by introducing HyperNeutrosophic Decision Making and SuperHyperNeutrosophic Decision Making, which are detailed in the following sections.

Definition 3.22 (Neutrosophic Decision Making (NDM))**.** [283] Neutrosophic Decision Making (NDM) is a multi-criteria decision-making framework that integrates neutrosophic logic, probability, and set theory to handle uncertainty, indeterminacy, and conflicting information.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of m alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of n criteria. Each alternative A_i is evaluated against each criterion C_j using a single-valued neutrosophic set (SVNS), represented as:

$$S_{ij} = \big(T_{ij}, I_{ij}, F_{ij}\big),$$

where:

- T_{ij} is the degree of truth that A_i satisfies C_j,
- I_{ij} is the degree of indeterminacy about A_i satisfying C_j,
- F_{ij} is the degree of falsity that A_i satisfies C_j.

These components satisfy the condition:

$$0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3, \quad \forall i, j.$$

Decision Matrix. The evaluations are organized into a decision matrix S:

$$S = \begin{bmatrix} S_{11} & S_{12} & \ldots & S_{1n} \\ S_{21} & S_{22} & \ldots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m1} & S_{m2} & \ldots & S_{mn} \end{bmatrix},$$

where each entry S_{ij} is a triple $\big(T_{ij}, I_{ij}, F_{ij}\big)$.

Weight Vector. Each criterion C_j is assigned a weight $w_j \in [0,1]$, representing its relative importance. The weights satisfy the normalization condition:

$$\sum_{j=1}^{n} w_j = 1.$$

Steps for Neutrosophic Decision Making. The steps for neutrosophic decision making are as follows:

1. Construct the decision matrix S using expert opinions or data, populating it with neutrosophic evaluations S_{ij}.
2. Identify the Ideal Neutrosophic Positive Solution (INPS) and Ideal Neutrosophic Negative Solution (INNS):
   - The INPS is given by:
   $$S^+ = \left(T_j^+, I_j^+, F_j^+\right), \quad T_j^+ = \max_i T_{ij},\ I_j^+ = \min_i I_{ij},\ F_j^+ = \min_i F_{ij}.$$
   - The INNS is given by:
   $$S^- = \left(T_j^-, I_j^-, F_j^-\right), \quad T_j^- = \min_i T_{ij},\ I_j^- = \max_i I_{ij},\ F_j^- = \max_i F_{ij}.$$
3. Compute the neutrosophic distance of each alternative A_i from both INPS and INNS using a distance metric such as Hamming or Euclidean distance.
4. Calculate the relative closeness of each alternative to INPS:
   $$RC_i = \frac{D_i^-}{D_i^+ + D_i^-},$$
   where D_i^+ and D_i^- are the distances of A_i from INPS and INNS, respectively.
5. Rank the alternatives based on their relative closeness RC_i. The alternative with the highest RC_i is the most preferred.

Definition 3.23 (HyperNeutrosophic Decision Making (HNDM))**.** HyperNeutrosophic Decision Making (HNDM) is an extension of Neutrosophic Decision Making (NDM) that incorporates a set of evaluations for each criterion, allowing for multiple perspectives or uncertain inputs to be represented simultaneously.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of m alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of n criteria. For each pair (A_i, C_j), the evaluation is represented as a set of triples:

$$H_{ij} = \left\{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\right\},$$

where:

- $T_{ij}^k \in [0,1]$ is the degree of truth for the k-th evaluation,
- $I_{ij}^k \in [0,1]$ is the degree of indeterminacy for the k-th evaluation,
- $F_{ij}^k \in [0,1]$ is the degree of falsity for the k-th evaluation,

and $\mathcal{K}_{ij}$ is the index set of available evaluations for (A_i, C_j). Each triple satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3, \quad \forall k \in \mathcal{K}_{ij}.$$

Steps for HNDM.

1. Construct the HyperNeutrosophic Decision Matrix H, where each entry H_{ij} is a set of evaluations.
2. Define the Ideal HyperNeutrosophic Positive Solution (IHNPS) and the Ideal HyperNeutrosophic Negative Solution (IHNNS) for each criterion C_j:

$$H_j^+ = \big\{ \max_k T_{ij}^k, \min_k I_{ij}^k, \min_k F_{ij}^k \big\},$$

$$H_j^- = \big\{ \min_k T_{ij}^k, \max_k I_{ij}^k, \max_k F_{ij}^k \big\}.$$

3. Compute the HyperNeutrosophic Distance for each alternative A_i from IHNPS and IHNNS.
4. Calculate the relative closeness to IHNPS for each alternative:

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-}.$$

5. Rank the alternatives based on RC_i.

Example 3.24 (Scenario: Medical Diagnosis and Treatment Plan Selection)**.** (cf. [167, 401, 433])

Problem: A patient requires a treatment plan for cancer (cf. [446]), and multiple diagnostic tests (cf. [114]) and expert opinions provide different perspectives on each option's effectiveness.

Setup:

- *Alternatives (A):* A_1: Chemotherapy [116], A_2: Radiation Therapy [57], A_3: Surgery [265].
- *Criteria (C):* C_1: Effectiveness, C_2: Side Effects, C_3: Cost.
- *Evaluations:* For each alternative A_i and criterion C_j, multiple evaluations are collected:

$$H_{ij} = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \text{ represents a test or expert opinion}\}.$$

  Example: For C_1 (Effectiveness of A_1), the results from three sources (e.g., imaging, biopsy [280], and expert review) could be:

$$H_{11} = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1), (0.85, 0.1, 0.05)\}.$$

Decision Process:

(1) Construct the HyperNeutrosophic Decision Matrix H with evaluations for each (A_i, C_j).

(2) Define the Ideal HyperNeutrosophic Positive Solution (IHNPS) to maximize effectiveness and minimize side effects and cost:

$$H_j^+ = \big\{ \max_k T_{ij}^k, \min_k I_{ij}^k, \min_k F_{ij}^k \big\}.$$

(3) Compute distances of each alternative A_i from IHNPS and Ideal HyperNeutrosophic Negative Solution (IHNNS).

(4) Rank the alternatives based on relative closeness to IHNPS.

Outcome: The treatment plan with the best balance of effectiveness, minimal side effects, and cost is chosen.

Definition 3.25 (n-SuperHyperNeutrosophic Decision Making (n-SHNDM))**.** n-SuperHyperNeutrosophic Decision Making (n-SHNDM) is a hierarchical generalization of HyperNeutrosophic Decision Making (HNDM), where the evaluations for each criterion are represented as nested sets of neutrosophic evaluations. This framework introduces n-fold nested hyperstructures derived from Neutrosophic Probability Distributions (NPDs), allowing for deeper levels of uncertainty, multi-layered inputs, and more complex modeling of preferences and criteria under uncertainty.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria. For each pair (A_i, C_j), the evaluation is represented as:

$$H_{ij}^{(n)} = \mathcal{P}^n\big([0,1]^3\big),$$

where:

- $n = 0$ reduces to Neutrosophic Decision Making, i.e., a single (T, I, F)-triple.
- $n = 1$ corresponds to HyperNeutrosophic Decision Making, i.e., a set of (T, I, F)-triples.
- $n > 1$ represents n-fold nested power sets of neutrosophic evaluations, capturing multiple layers of uncertainty and complexity.

Steps for n-SHNDM.

(1) Construct the n-SuperHyperNeutrosophic Decision Matrix $H^{(n)}$, where each entry $H_{ij}^{(n)}$ is an n-fold nested set of neutrosophic evaluations.

(2) Define the Ideal n-SuperHyperNeutrosophic Positive Solution (n-SHNPS) and the Ideal n-SuperHyperNeutrosophic Negative Solution (n-SHNNS):

$$H_j^+ = \max\big(H_{ij}^{(n)}\big), \quad H_j^- = \min\big(H_{ij}^{(n)}\big),$$

using suitable aggregation operators that account for the nested neutrosophic structure.

(3) Compute the n-SuperHyperNeutrosophic Distance for each alternative A_i from n-SHNPS and n-SHNNS.

(4) Calculate the relative closeness to n-SHNPS for each alternative:

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-}.$$

(5) Rank the alternatives based on RC_i.

Theorem 3.26 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Decision Making possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.27 (Reduction Property)**.** *For $n = 0$, n-SHNDM reduces to standard Neutrosophic Decision Making.*

Proof. When $n = 0$, no power set operations are applied. Hence, each evaluation $H_{ij}^{(0)}$ is simply a neutrosophic triple (T, I, F) directly representing the degree of truth, indeterminacy, and falsity for the criterion C_j applied to alternative A_i. This coincides with the standard Neutrosophic Decision Making model, proving the reduction property. □

Theorem 3.28 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, the evaluations in n-SHNDM are strictly more complex (hierarchically) than those in $(n-1)$-SHNDM.*

Proof. In $(n-1)$-SHNDM, each evaluation $H_{ij}^{(n-1)}$ is an $(n-1)$-fold nested power set of neutrosophic triples. To obtain n-SHNDM, we apply one additional power set operation:

$$H_{ij}^{(n)} = \mathcal{P}(H_{ij}^{(n-1)}).$$

The power set operation generates a strictly larger and more complex family of subsets. Thus, n-SHNDM inherently introduces a higher level of complexity than $(n-1)$-SHNDM. □

Theorem 3.29 (Existence of Ideal Solutions)**.** *For any n-SHNDM problem, the Ideal n-SuperHyperNeutrosophic Positive Solution (n-SHNPS) and the Ideal n-SuperHyperNeutrosophic Negative Solution (n-SHNNS) exist and are well-defined.*

Proof. Each $H_{ij}^{(n)}$ is a non-empty n-fold nested power set of neutrosophic triples. By the properties of power sets, these structures are always non-empty. Applying the max and min operations to these nested sets using appropriate neutrosophic aggregation operators yields non-empty results. Therefore, H_j^+ and H_j^- are always well-defined. Since this holds for all criteria C_j, the sets $\{H_j^+\}$ and $\{H_j^-\}$ form the n-SHNPS and n-SHNNS respectively, ensuring their existence. □

Theorem 3.30 (Continuity under Parameter Perturbations)**.** *If the neutrosophic evaluations (T, I, F) within the nested sets vary continuously with respect to parameter changes (e.g., weighting factors or external conditions), then the resulting rankings in n-SHNDM vary continuously as well.*

Proof. The decision-making process in n-SHNDM involves:

$$H_{ij}^{(n)} = \mathcal{P}^n([0,1]^3),$$

and subsequent aggregation steps (max, min, distance computations, and relative closeness) are compositions of continuous operations over compact sets in $[0,1]^3$. As continuous transformations of continuous parameterized inputs, the outputs (rankings) must also be continuous. Small perturbations in the input values yield small changes in the evaluated sets and, consequently, in the derived ranking, proving continuity. □

Theorem 3.31 (Monotonicity with Respect to Truth Degrees)**.** *Consider two alternatives A_p and A_q and a single criterion C_j. If every neutrosophic triple in $H_{pj}^{(n)}$ has greater or equal truth degrees T than the corresponding triples in $H_{qj}^{(n)}$ (with other factors equal), then A_p cannot rank lower than A_q.*

Proof. The decision-making framework ranks alternatives based on their relative closeness to the n-SHNPS and n-SHNNS, which ultimately depends on the values of T, I, and F within the nested sets. If $H_{pj}^{(n)}$ uniformly dominates $H_{qj}^{(n)}$ in truth (and all else is equal), the aggregated evaluation for A_p must be at least as favorable as that for A_q, preserving or improving A_p's relative closeness and thus ensuring A_p does not rank below A_q. □

Similarly, HyperFuzzy Decision Making, HyperPlithogenic Decision Making, SuperHyperFuzzy Decision Making, and SuperHyperPlithogenic Decision Making can also be defined.

3.4. *HyperNeutrosophic Language*

This subsection examines the concept of HyperNeutrosophic Language. Various linguistic frameworks, including Formal Languages [152, 190, 198, 315, 329] and Natural Languages [65, 98, 257], have been extensively explored in the literature. With advancements in fields such as machine learning, these studies have garnered significant attention, particularly in areas like Natural Language Processing (cf. [65, 98, 101, 168, 241, 257, 258, 301, 421, 432, 438, 444]). Technologies such as ChatGPT also incorporate these concepts in their applications ([7, 55, 133, 189, 212, 248]). Moreover, related ideas like HyperLanguage [69, 70, 132, 135, 144] and SuperHyperLanguage [135, 144] are well-documented and serve as foundational concepts for further exploration.

Related concepts, such as Fuzzy Language [108,109,305], Neutrosophic Language [144], and Plithogenic Language [144], have also been explored. In this context, we propose extending these frameworks to define HyperNeutrosophic Language and SuperHyperNeutrosophic Language. Their definitions and related concepts are outlined below.

Definition 3.32 (Neutrosophic Language)**.** [144] Let Σ be a finite alphabet. A *Neutrosophic Language* over Σ^* is a function:

$$N : \Sigma^* \to [0,1]^3,$$

where for each word $w \in \Sigma^*$, $N(w) = (T(w), I(w), F(w))$ with $T(w), I(w), F(w) \in [0,1]$ and

$$0 \leq T(w) + I(w) + F(w) \leq 3.$$

Here:

- $T(w)$ represents the *truth-membership degree* of w.
- $I(w)$ represents the *indeterminacy-membership degree* of w.
- $F(w)$ represents the *falsity-membership degree* of w.

A Neutrosophic Language generalizes the notion of membership beyond the single membership function of a fuzzy language by explicitly incorporating degrees of truth, indeterminacy, and falsity.

Example 3.33 (Applications of Neutrosophic Language in Real Life)**.** We present two practical examples of Neutrosophic Language.

1. Sentiment Analysis in Product Reviews: Consider a product review system [8,66] where $\Sigma = \{\text{excellent}, \text{good}, \text{average}, \text{poor}\}$. For each word w in the review, the Neutrosophic Language $N(w)$ assigns:

$$N(\text{excellent}) = (0.9, 0.05, 0.05), \quad N(\text{average}) = (0.5, 0.4, 0.1).$$

Here, $T(w)$ indicates the degree of positive sentiment, $I(w)$ represents uncertainty (e.g., mixed opinions), and $F(w)$ reflects negative sentiment. By incorporating these components, the system can perform nuanced analysis, considering both explicit sentiment and inherent ambiguities.

2. Medical Diagnosis in Healthcare: In a healthcare diagnostic system [348], let $\Sigma = \{\text{healthy}, \text{ill}, \text{uncertain}\}$. For a patient record word w, the Neutrosophic Language $N(w)$ assigns:

$$N(\text{healthy}) = (0.85, 0.1, 0.05), \quad N(\text{ill}) = (0.3, 0.5, 0.2).$$

Here, $T(w)$ quantifies the likelihood of being healthy, $I(w)$ captures uncertainty in the diagnosis, and $F(w)$ indicates the likelihood of being ill. This provides a structured framework for handling cases with incomplete or conflicting data.

Definition 3.34 (HyperNeutrosophic Language)**.** Let Σ be a finite alphabet. A *HyperNeutrosophic Language* over Σ^* is a function:

$$H : \Sigma^* \to \mathcal{P}([0,1]^3),$$

where for each word $w \in \Sigma^*$, $H(w) \subseteq [0,1]^3$ represents a set of neutrosophic triples:

$$H(w) = \{(T_k(w), I_k(w), F_k(w)) \mid k \in \mathcal{K}_w\},$$

where:

- $T_k(w), I_k(w), F_k(w) \in [0,1]$ are the k-th truth, indeterminacy, and falsity membership degrees, respectively.
- $\mathcal{K}_w$ is the index set of evaluations associated with w.
- Each triple satisfies:

$$0 \leq T_k(w) + I_k(w) + F_k(w) \leq 3.$$

The HyperNeutrosophic Language generalizes the Neutrosophic Language by allowing multiple evaluations for each word, enabling the representation of uncertain or conflicting linguistic data from different sources.

Example 3.35 (Applications of HyperNeutrosophic Language in Real Life)**.** This section presents two practical examples of HyperNeutrosophic Language.

1. Machine Translation with Multiple Evaluations: Consider a machine translation system [435] evaluated by multiple experts. Let $\Sigma = \{\text{hello}, \text{goodbye}\}$. For each word w, the HyperNeutrosophic Language $H(w)$ assigns:

$$H(\text{hello}) = \{(0.95, 0.03, 0.02), (0.90, 0.05, 0.05)\}, \quad H(\text{goodbye}) = \{(0.85, 0.1, 0.05), (0.8, 0.15, 0.05)\}.$$

Each triple corresponds to a different expert's evaluation, capturing variability in the perceived accuracy and reliability of translations.

2. Crowdsourced Content Moderation: In a content moderation system [406], let $\Sigma = \{\text{safe}, \text{offensive}\}$. For each word w, the HyperNeutrosophic Language $H(w)$ assigns:

$$H(\text{safe}) = \{(0.8, 0.1, 0.1), (0.85, 0.05, 0.1)\}, \quad H(\text{offensive}) = \{(0.2, 0.6, 0.2), (0.3, 0.5, 0.2)\}.$$

These evaluations reflect judgments from multiple moderators, allowing the system to incorporate diverse perspectives and handle ambiguous cases effectively.

Definition 3.36 (n-SuperHyperNeutrosophic Language)**.** Let Σ be a finite alphabet. An *n-SuperHyperNeutrosophic Language* over Σ^* is a function:

$$H^{(n)} : \Sigma^* \to \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ represents the n-th nested power set of $[0,1]^3$. For each word $w \in \Sigma^*$, the evaluation is recursively defined as:

$$H^{(n)}(w) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(H^{(n-1)}(w)), & \text{if } n \geq 1. \end{cases}$$

In this framework:

- $n = 0$ corresponds to the basic Neutrosophic Language.
- $n = 1$ corresponds to the HyperNeutrosophic Language.
- $n > 1$ represents higher-order structures, where each level of nesting allows for increasingly complex representations of uncertainty and multi-layered evaluations.

Theorem 3.37 (Relation of an n-SuperHyperNeutrosophic Set)**.** *An n-SuperHyperNeutrosophic Language inherently exhibits the structure of an n-SuperHyperNeutrosophic Set.*

Proof. The statement follows directly and is self-evident. □

Theorem 3.38 (Generalization Property)**.** *For any $n \geq 0$, an n-SuperHyperNeutrosophic Language (n-SHNL) generalizes the concepts of Neutrosophic Language (NL) and HyperNeutrosophic Language (HNL). Specifically:*

(1) *For $n = 0$, n-SHNL reduces to a Neutrosophic Language.*
(2) *For $n = 1$, n-SHNL coincides with a HyperNeutrosophic Language.*
(3) *For $n > 1$, n-SHNL provides a hierarchical nesting of neutrosophic evaluations, extending beyond both NL and HNL.*

Proof. By Definition 3.36, we have:

$$H^{(n)} : \Sigma^* \to \mathcal{P}^n([0,1]^3).$$

Case $n = 0$: If $n = 0$, by definition:

$$H^{(0)}(w) = [0,1]^3,$$

which means for each word w, we assign a single triple $(T(w), I(w), F(w)) \in [0,1]^3$. This matches exactly the definition of a Neutrosophic Language (Definition 3.32), where each word w maps to a single neutrosophic triple. Thus, 0-SHNL = NL.

Case $n = 1$: If $n = 1$:

$$H^{(1)}(w) = \mathcal{P}(H^{(0)}(w)) = \mathcal{P}([0,1]^3).$$

This means for each word w, we assign a set of triples, each triple being a point in $[0,1]^3$. By Definition 3.34, this is exactly the definition of a HyperNeutrosophic Language. Hence, 1-SHNL = HNL.

Case $n > 1$: For $n > 1$,

$$H^{(n)}(w) = \mathcal{P}(H^{(n-1)}(w)).$$

Since $H^{(n-1)}(w)$ itself is formed by $(n-1)$-fold nesting of power sets of $[0,1]^3$, taking another power set increases the complexity and the hierarchy of the uncertainty representation. Thus, for $n > 1$, $H^{(n)}$ is a strict extension, allowing for n-layered uncertainty and complexity not captured by NL or HNL.

Combining these results, we see that n-SHNL generalizes both NL (at $n = 0$) and HNL (at $n = 1$), and provides greater complexity for $n > 1$. □

Theorem 3.39 (Reduction to Classical Languages)**.** *If for all words $w \in \Sigma^*$ and for all levels of nesting, the indeterminacy I and falsity F components vanish (i.e., $I(w) = 0$ and $F(w) = 0$ at the base level and consequently at all higher levels of nesting), then an n-SuperHyperNeutrosophic Language reduces to a classical crisp language membership function.*

Proof. Suppose for each $w \in \Sigma^*$ and at all levels of n-nesting, we have:

$$I(w) = 0, \quad F(w) = 0.$$

Under these conditions, every neutrosophic triple simplifies to:

$$(T(w), 0, 0),$$

where $T(w) \in [0,1]$.

At $n = 0$ (the Neutrosophic Language level), $N(w) = (T(w), 0, 0)$ simply becomes a scalar membership in $[0,1]$. If $T(w)$ is further restricted to be in $\{0,1\}$, then it becomes a classical characteristic function defining a crisp language:

$$L = \{w \mid T(w) = 1\}.$$

For $n = 1$ (the HyperNeutrosophic level) or any $n > 1$, the nested sets of triples would also collapse to sets of $(T, 0, 0)$ values. Without indeterminacy or falsity, and assuming a crisp threshold, the entire nested structure does not add complexity. Ultimately, it boils down to a deterministic assignment, indistinguishable from a classical language membership if $T(w)$ is binary.

Thus, in the absence of indeterminacy and falsity, n-SHNL reduces to a classical language with crisp membership, confirming the claim. □

Theorem 3.40 (Monotonicity of Complexity)**.** *Let $H^{(n)}$ be an n-SuperHyperNeutrosophic Language. As n increases, the representational capacity and complexity of $H^{(n)}$ is non-decreasing. In particular, for all $m > n$, $H^{(m)}$ can represent at least as complex a structure as $H^{(n)}$.*

Proof. By definition:

$$H^{(n)}(w) \in \mathcal{P}^n([0,1]^3),$$

and

$$H^{(m)}(w) \in \mathcal{P}^m([0,1]^3).$$

Since $\mathcal{P}^m([0,1]^3)$ represents an m-fold power set of $[0,1]^3$, and $m > n$, we have:

$$\mathcal{P}^n([0,1]^3) \subseteq \mathcal{P}^m([0,1]^3).$$

This inclusion follows from the construction: each level of power set strictly expands or maintains the set of representable structures. No operation reduces complexity; at worst, it preserves it.

Thus, any neutrosophic evaluation representable at level n can be embedded into the representation at level $m > n$. Consequently, the complexity and representational capacity is non-decreasing as n grows, establishing monotonicity. □

Theorem 3.41 (Continuity of Representation)**.** *If the base-level neutrosophic evaluations* $(T(w), I(w), F(w))$ *vary continuously in* $[0,1]^3$ *for each word* w*, then the resulting n-SuperHyperNeutrosophic Language* $H^{(n)}$ *changes continuously with respect to these variations at all nesting levels.*

Proof. At $n = 0$, the mapping $N(w) = (T(w), I(w), F(w))$ is trivially continuous as each component is a direct evaluation in $[0,1]$.

For $n = 1$, $H^{(1)}(w) = \mathcal{P}(N(w))$. The power set operator at this level represents sets of neutrosophic triples. Small continuous changes in $(T(w), I(w), F(w))$ at the base level reflect continuous changes in these sets (e.g., if sets are constructed by applying continuous selection criteria or aggregation functions).

By induction, suppose continuity holds for $H^{(n-1)}(w)$. The construction of $H^{(n)}(w) = \mathcal{P}(H^{(n-1)}(w))$ involves set operations and possibly aggregation operators that are continuous with respect to the underlying neutrosophic triples. Since composition of continuous operations remains continuous, $H^{(n)}(w)$ also varies continuously as $(T(w), I(w), F(w))$ at the base level change.

Hence, at every level of nesting, continuity is preserved. This proves that $H^{(n)}$ is a continuous extension of the base-level neutrosophic evaluations. □

Similarly, HyperFuzzy Language, HyperPlithogenic Language, SuperHyperFuzzy Language, and SuperHyperPlithogenic Language can also be defined.

Question 3.42. Can concepts such as SuperHyperPlithogenic HyperLanguage and SuperHyperPlithogenic SuperHyperLanguage, along with their related frameworks, be formally defined? What are their defining characteristics? Furthermore, is it possible to apply these concepts to natural language processing?

Question 3.43. Related concepts such as natural language generation [211, 238, 316], natural language understanding [36, 117, 303, 441], natural language inference [73, 94, 102, 232], and natural language interface [38, 194] have been extensively studied.

Is it possible to extend these ideas to define and explore concepts like Natural HyperLanguage, Natural SuperHyperLanguage, Natural HyperPlithogenic Language, and Natural SuperHyperPlithogenic Language? Could these extensions have practical applications in certain fields?

3.5. *HyperNeutrosophic Topological Space*

A Topological Space is a set of points with a topology, defining open sets to study continuity, convergence, and neighborhood relationships [30, 58, 89, 97, 204, 226]. Extended concepts such as Fuzzy Topological Space [91, 105, 160, 434], Neutrosophic Topological Space [31, 229], and Plithogenic Topological Space [9] are well-known. This paper further extends these to define HyperNeutrosophic Topological Space and SuperHyperNeutrosophic Topological Space.

Definition 3.44 (Neutrosophic Topological Space)**.** Let X be a non-empty set. A *Neutrosophic Topology (NT)* on X is a family τ of neutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau$, where:

$$N_0 = \{(x, 0, 0, 0) \mid x \in X\}, \quad N_1 = \{(x, 1, 1, 1) \mid x \in X\}.$$

(2) The intersection of any two neutrosophic open sets is also neutrosophic open:

$$G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau.$$

(3) The union of any collection of neutrosophic open sets is neutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau\} \implies \bigcup_{i \in J} G_i \in \tau.$$

In this case, the pair (X, τ) is called a *Neutrosophic Topological Space (NTS)*, and any set in τ is referred to as a *neutrosophic open set (NOS)*(cf. [205, 312, 336]).

Definition 3.45 (HyperNeutrosophic Topological Space)**.** Let X be a non-empty set. A *HyperNeutrosophic Topology (HNT)* on X is a family τ_H of hyperneutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau_H$, where:

$$N_0 = \{(x, \{(0,0,0)\}) \mid x \in X\}, \quad N_1 = \{(x, \{(1,1,1)\}) \mid x \in X\}.$$

(2) The intersection of any two hyperneutrosophic open sets is also hyperneutrosophic open:

$$G_1, G_2 \in \tau_H \implies G_1 \cap G_2 \in \tau_H.$$

(3) The union of any collection of hyperneutrosophic open sets is hyperneutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau_H\} \implies \bigcup_{i \in J} G_i \in \tau_H.$$

Each hyperneutrosophic set is defined as:

$$H(x) = \{(T_k(x), I_k(x), F_k(x)) \mid k \in \mathcal{K}_x\},$$

where:

- $T_k(x), I_k(x), F_k(x) \in [0,1]$ are the k-th truth, indeterminacy, and falsity degrees, respectively.
- $\mathcal{K}_x$ is the index set of evaluations for x.
- Each triple satisfies:

$$0 \leq T_k(x) + I_k(x) + F_k(x) \leq 3.$$

In this case, the pair (X, τ_H) is called a *HyperNeutrosophic Topological Space (H-NTS)*, and any set in τ_H is referred to as a *hyperneutrosophic open set*.

Definition 3.46 (n-SuperHyperNeutrosophic Topological Space)**.** Let X be a non-empty set. An *n-SuperHyperNeutrosophic Topology (n-S-HNT)* on X is a family $\tau_H^{(n)}$ of n-nested hyperneutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau_H^{(n)}$, where:

$$N_0 = \{(x, \mathcal{P}^n(\{(0,0,0)\})) \mid x \in X\}, \quad N_1 = \{(x, \mathcal{P}^n(\{(1,1,1)\})) \mid x \in X\}.$$

(2) The intersection of any two n-superhyperneutrosophic open sets is also n-superhyperneutrosophic open:

$$G_1, G_2 \in \tau_H^{(n)} \implies G_1 \cap G_2 \in \tau_H^{(n)}.$$

(3) The union of any collection of n-superhyperneutrosophic open sets is n-superhyperneutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau_H^{(n)}\} \implies \bigcup_{i \in J} G_i \in \tau_H^{(n)}.$$

Each n-superhyperneutrosophic set is recursively defined as:

$$H^{(n)}(x) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(H^{(n-1)}(x)), & \text{if } n \geq 1. \end{cases}$$

In this case, the pair $(X, \tau_H^{(n)})$ is called an *n-SuperHyperNeutrosophic Topological Space (n-S-HNTS)*, and any set in $\tau_H^{(n)}$ is referred to as an n-superhyperneutrosophic open set.

Theorem 3.47 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Topological Space possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.48 (Reduction to Neutrosophic Topological Spaces)**.** *If $n = 0$, then an n-SuperHyperNeutrosophic Topological Space (n-S-HNTS) reduces to a standard Neutrosophic Topological Space (NTS).*

Proof. For $n = 0$, we have:

$$H^{(0)}(x) = [0,1]^3,$$

representing a single neutrosophic triple $(T(x), I(x), F(x))$ with $0 \leq T(x) + I(x) + F(x) \leq 3$. No power set nesting is involved at this level, so the topology $\tau_H^{(0)}$ consists solely of neutrosophic subsets of X satisfying the usual neutrosophic topological axioms. Thus, $(X, \tau_H^{(0)})$ is precisely an NTS. □

Theorem 3.49 (Increasing Hierarchical Complexity)**.** *For every $n \geq 1$, an n-S-HNTS possesses strictly greater structural complexity than an $(n-1)$-S-HNTS.*

Proof. By construction, moving from level $(n-1)$ to n applies the power set operation $\mathcal{P}$:

$$H^{(n)}(x) = \mathcal{P}(H^{(n-1)}(x)).$$

The power set of a non-empty set is always strictly larger, introducing exponentially more subsets. Consequently, each step in this hierarchy adds a new layer of complexity, creating an increasingly intricate family of open sets. Hence, n-S-HNTS is more complex than $(n-1)$-S-HNTS in terms of the richness and combinatorial depth of its topological structure. □

Theorem 3.50 (Inductive Construction of n-S-HNTS)**.** *Suppose $(X, \tau_H^{(n-1)})$ is an $(n-1)$-SuperHyperNeutrosophic Topological Space. Define*

$$\tau_H^{(n)} = \{\mathcal{P}(G') \mid G' \in \tau_H^{(n-1)}\}.$$

Then $(X, \tau_H^{(n)})$ is an n-SuperHyperNeutrosophic Topological Space.

Proof. We use induction on n:

Base case ($n = 0$)*:* An 0-S-HNTS is just an NTS, which satisfies the neutrosophic topological axioms by definition.

Inductive step: Assume $(X, \tau_H^{(n-1)})$ is an $(n-1)$-S-HNTS. In particular:

(1) $N_0, N_1 \in \tau_H^{(n-1)}$.

(2) Finite intersections and arbitrary unions of sets in $\tau_H^{(n-1)}$ lie in $\tau_H^{(n-1)}$.

Define $\tau_H^{(n)} = \{\mathcal{P}(G') \mid G' \in \tau_H^{(n-1)}\}$.

- Since $N_0, N_1 \in \tau_H^{(n-1)}$, we have $\mathcal{P}(N_0), \mathcal{P}(N_1) \in \tau_H^{(n)}$. - For $G_1', G_2' \in \tau_H^{(n-1)}$, their intersection $G_1' \cap G_2'$ lies in $\tau_H^{(n-1)}$. Thus:

$$\mathcal{P}(G_1') \cap \mathcal{P}(G_2') = \mathcal{P}(G_1' \cap G_2') \in \tau_H^{(n)}.$$

- For any collection $\{G_i'\}_{i \in J} \subseteq \tau_H^{(n-1)}$:

$$\bigcup_{i \in J} G_i' \in \tau_H^{(n-1)} \implies \bigcup_{i \in J} \mathcal{P}(G_i') = \mathcal{P}\left(\bigcup_{i \in J} G_i'\right) \in \tau_H^{(n)}.$$

These verifications show that $\tau_H^{(n)}$ inherits the topological axioms from $\tau_H^{(n-1)}$. Therefore, $(X, \tau_H^{(n)})$ is indeed an n-S-HNTS. □

Definition 3.51 (Homomorphism)**.** (cf. [126,328]) A *homomorphism* is a structure-preserving map between two algebraic structures of the same type. For example, a function $f : A \to B$ is a homomorphism between two groups $(A, \cdot)$ and $(B, *)$ if for all $x, y \in A$:

$$f(x \cdot y) = f(x) * f(y).$$

Theorem 3.52 (Reduction Homomorphism)**.** *There exists a natural reduction mapping* $\rho_n : \tau_H^{(n)} \to \tau_H^{(n-1)}$ *that, given an* n*-S-HNTS, recovers the* $(n-1)$*-S-HNTS structure without violating the topological axioms.*

Proof. Define ρ_n to map each $G \in \tau_H^{(n)}$, where $G = \mathcal{P}(G')$ for some $G' \in \tau_H^{(n-1)}$, to the corresponding G'. This mapping is well-defined because every set at level n arises from taking a power set of a set at level $(n-1)$.

For any two sets $G_1, G_2 \in \tau_H^{(n)}$:

$$\rho_n(G_1 \cap G_2) = \rho_n(\mathcal{P}(G_1') \cap \mathcal{P}(G_2')) = \rho_n(\mathcal{P}(G_1' \cap G_2')) = G_1' \cap G_2' = \rho_n(G_1) \cap \rho_n(G_2).$$

For any collection $\{G_i\}_{i \in J} \subseteq \tau_H^{(n)}$:

$$\rho_n\left(\bigcup_{i \in J} G_i\right) = \rho_n\left(\mathcal{P}\left(\bigcup_{i \in J} G_i'\right)\right) = \bigcup_{i \in J} G_i' = \bigcup_{i \in J} \rho_n(G_i).$$

Thus, ρ_n respects both finite intersections and arbitrary unions, preserving the topological structure. This shows that ρ_n acts as a natural homomorphism, reducing an n-S-HNTS back to $(n-1)$-S-HNTS while maintaining all topological properties. □

Theorem 3.53 (Boundedness of the Hierarchy)**.** *At each level n, the sets remain within $\mathcal{P}^n([0,1]^3)$, ensuring that the hierarchy of n-S-HNTS is bounded within the neutrosophic cube structure.*

Proof. Since each n-S-HNTS is constructed by repeatedly applying the power set operation to subsets of $[0,1]^3$, no element outside $[0,1]^3$ ever appears. The neutrosophic triples, and their nested power sets, all lie within a bounded domain. Hence, at any level n, the sets remain subsets of $\mathcal{P}^n([0,1]^3)$, ensuring that the entire hierarchy is contained within a finite, well-defined structure. □

Theorem 3.54 (Continuity under Nesting)**.** *If the neutrosophic values (T, I, F) vary continuously within $[0,1]^3$, then the induced topologies at all levels n change continuously with respect to these values.*

Proof. The construction of $\tau_H^{(n)}$ from $\tau_H^{(n-1)}$ relies on the power set operation, which is purely set-theoretic and does not introduce discontinuities. Since the underlying neutrosophic triples are themselves defined within the continuous interval $[0,1]^3$, small perturbations in T, I, or F values affect the membership of sets in a continuous manner. Consequently, the resulting topologies at each level n vary continuously under small changes in the underlying neutrosophic values. □

Similarly, HyperFuzzy Topological Space, HyperPlithogenic Topological Space, SuperHyperFuzzy Topological Space, and SuperHyperPlithogenic Topological Space can also be defined.

3.6. *HyperNeutrosophic K-means Clustering*

K-means Clustering is an unsupervised machine learning algorithm that partitions data into k clusters by minimizing intra-cluster variance [191,203,209,222,342]. Extended concepts such as Fuzzy K-means Clustering [162,235,240,242,404,436] and Neutrosophic K-means Clustering [302, 319] are well-known. This paper further extends these ideas to HyperNeutrosophic K-means Clustering. The definition is provided below.

Definition 3.55 (Neutrosophic K-means Clustering)**.** [302,319] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a dataset, where each $x_i \in \mathbb{R}^d$. The goal of the Neutrosophic K-means algorithm is to partition X into k clusters $C_1, C_2, \ldots, C_k$ by minimizing the objective function:

$$J(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \mu_{C_j}(x_i)^m \|x_i - v_j\|^2,$$

where:

- $\mu_{C_j}(x_i)$ is the neutrosophic membership of x_i in cluster C_j, given as:

$$\mu_{C_j}(x_i) = (T_{ij}, I_{ij}, F_{ij}),$$

  with $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$ satisfying $0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3$.
- $V = \{v_1, v_2, \ldots, v_k\}$ are the centroids of the clusters.
- $m > 1$ is a weighting exponent.

The algorithm iterates as follows:

1. Update the membership degrees:

$$\mu_{C_j}(x_i) = \left(\frac{\|x_i - v_j\|^{-2/(m-1)}}{\sum_{l=1}^{k} \|x_i - v_l\|^{-2/(m-1)}} \right).$$

2. Update the cluster centroids:

$$v_j = \frac{\sum_{x_i \in C_j} \mu_{C_j}(x_i)^m x_i}{\sum_{x_i \in C_j} \mu_{C_j}(x_i)^m}.$$

3. Repeat until convergence of $J(P, V)$.

Definition 3.56 (HyperNeutrosophic K-means Clustering)**.** The *HyperNeutrosophic K-means* extends the Neutrosophic K-means by allowing multiple neutrosophic membership evaluations for each data point. For each $x_i \in X$, the membership function is:

$$\mu_{C_j}(x_i) = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\},$$

where $\mathcal{K}_{ij}$ is the index set of evaluations. Each $(T_{ij}^k, I_{ij}^k, F_{ij}^k)$ satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3.$$

The objective function becomes:

$$J(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{k \in \mathcal{K}_{ij}} \mu_{C_j}^k(x_i)^m \right) \|x_i - v_j\|^2.$$

The steps for updating $\mu_{C_j}(x_i)$ and v_j follow similar iterative updates as in the Neutrosophic K-means, applied over all evaluations k.

Definition 3.57 (n-SuperHyperNeutrosophic K-means Clustering)**.** The *n-SuperHyperNeutrosophic K-means* generalizes HyperNeutrosophic K-means by recursively nesting the membership evaluations. For each $x_i \in X$, the membership function is:

$$\mu_{C_j}^{(n)}(x_i) = \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ denotes the n-th power set of the neutrosophic unit cube. For $n = 0$, $\mu_{C_j}^{(0)}(x_i)$ reduces to a single neutrosophic triple. For $n = 1$, $\mu_{C_j}^{(1)}(x_i)$ corresponds to a set of triples as in HyperNeutrosophic K-means.

The objective function is recursively defined as:

$$J^{(n)}(P,V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{S \in \mu_{C_j}^{(n)}(x_i)} S^m \|x_i - v_j\|^2 \right).$$

The updates for $\mu_{C_j}^{(n)}(x_i)$ and v_j are similarly extended iteratively to handle the nested structure of the membership function.

Theorem 3.58 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic K-means Clustering possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.59 (Reduction to Neutrosophic K-means Clustering)**.** *If $n = 0$, then n-SuperHyperNeutrosophic K-means clustering coincides with standard Neutrosophic K-means clustering.*

Proof. For $n = 0$, the membership function reduces to:

$$\mu_{C_j}^{(0)}(x_i) = [0,1]^3,$$

which corresponds to a single neutrosophic triple (T_{ij}, I_{ij}, F_{ij}). Under these conditions, the objective function

$$J^{(0)}(P,V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \mu_{C_j}(x_i)^m \|x_i - v_j\|^2$$

matches the standard Neutrosophic K-means objective. Consequently, centroid and membership updates follow the original Neutrosophic K-means algorithm. Thus, the 0-SuperHyperNeutrosophic K-means framework exactly replicates the standard Neutrosophic K-means approach. □

Theorem 3.60 (Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SuperHyperNeutrosophic K-means framework is strictly more complex in membership representation than the $(n-1)$-SuperHyperNeutrosophic K-means framework.*

Proof. The membership function at level n is defined by:

$$\mu_{C_j}^{(n)}(x_i) = \mathcal{P}(\mu_{C_j}^{(n-1)}(x_i)),$$

where $\mathcal{P}$ denotes the power set. Applying the power set operation increases the number of subsets, thereby expanding the complexity of the membership structure. Since the power set of a non-empty set is strictly larger than the original set, the complexity of $\mu_{C_j}^{(n)}(x_i)$ strictly surpasses that of $\mu_{C_j}^{(n-1)}(x_i)$. This exponential growth in complexity propagates throughout the clustering process, making each successive level n more involved than the previous one. □

Theorem 3.61 (Continuity of the Objective Function)**.** *For any $n \geq 0$, the objective function $J^{(n)}(P, V)$ in n-SuperHyperNeutrosophic K-means clustering is continuous with respect to membership values and centroids.*

Proof. The objective function at level n is:

$$J^{(n)}(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{S \in \mu_{C_j}^{(n)}(x_i)} S^m \|x_i - v_j\|^2 \right).$$

Each term $S^m \|x_i - v_j\|^2$ involves elementary continuous operations (exponentiation, multiplication, and distance calculation). The membership values S and the centroids v_j enter these operations in a continuous manner. Since summations and power set aggregations preserve continuity, the entire objective function remains continuous. □

Theorem 3.62 (Convergence to a Local Minimum)**.** *The n-SuperHyperNeutrosophic K-means algorithm converges to a local minimum of the objective function $J^{(n)}(P, V)$.*

Proof. The algorithm updates membership values and centroids iteratively. Each update is designed to reduce the objective function or keep it constant, ensuring that the sequence of objective values is monotonically non-increasing and bounded below by zero. By the monotone convergence property, the algorithm must converge to a point where no further reduction is possible, i.e., a local minimum of $J^{(n)}(P, V)$. □

Theorem 3.63 (Partitioning Stability)**.** *For any fixed $n \geq 0$, the partitions obtained by n-SuperHyperNeutrosophic K-means clustering are stable under small perturbations of the input data.*

Proof. Small changes in the input data x_i produce correspondingly small changes in $\|x_i - v_j\|$. Since membership values depend continuously on these distances and are bounded, slight input variations lead to marginal adjustments in membership values and centroids. Thus, the cluster memberships and resulting partitions remain close to their original configuration, ensuring stability and robustness against minor data perturbations. □

Theorem 3.64 (Scalability of Complexity)**.** *As n increases, the complexity of the n-SuperHyperNeutrosophic K-means algorithm grows exponentially with respect to the size of the membership sets.*

Proof. At level n, the membership structure involves $\mathcal{P}^n([0,1]^3)$, representing n-fold nested power sets. Each nesting layer multiplies the number of subsets, leading to an exponential increase in the cardinality of the membership representation. Consequently, computations involving membership aggregation, centroid updates, and objective function evaluations scale exponentially, reflecting a rapid growth in complexity as n increases. □

Theorem 3.65 (Generalization to Other Metrics)**.** *The properties established for n-SuperHyperNeutrosophic K-means remain valid if the Euclidean norm $\|x_i - v_j\|$ is replaced by any continuous distance metric.*

Proof. All continuity, convergence, and stability arguments rely on the continuity and boundedness of the operations involved. Replacing the Euclidean norm with any other continuous distance metric preserves these properties. Since the definitions of membership updates, objective function computations, and centroid calculations depend only on continuity and boundedness, all theoretical guarantees extend naturally to other continuous metrics. □

Similarly, HyperFuzzy K-means, HyperPlithogenic K-means, SuperHyperFuzzy K-means, and SuperHyperPlithogenic K-means can also be defined. Additionally, extensions to related concepts such as C-means [6, 10, 183] can also be considered.

3.7. *Neutrosophic TOPSIS*

TOPSIS is a multi-criteria decision-making method selecting the best alternative by proximity to ideal and anti-ideal solutions [92, 127, 281]. As extensions of TOPSIS, methods like Fuzzy TOPSIS [181, 184, 186, 274, 282, 337, 405, 462] and Plithogenic TOPSIS [5, 260] are well-known. Here, we introduce Neutrosophic TOPSIS [4, 28, 218, 231, 244, 272, 273, 297]. Its definitions and related concepts are outlined below.

Definition 3.66 (Neutrosophic TOPSIS)**.** (cf. [244, 272, 273, 297]) Neutrosophic TOPSIS is a multi-criteria decision-making method utilizing neutrosophic triples (T, I, F) for each criterion. Let $A = \{A_1, \ldots, A_m\}$ be alternatives and $C = \{C_1, \ldots, C_n\}$ be criteria. The steps are as follows:

(1) Construct the decision matrix $D_{ij} = (T_{ij}, I_{ij}, F_{ij})$, where $T, I, F \in [0, 1]$ and $T+I+F \leq 3$.

(2) Normalize and weight the matrix. Normalize D_{ij} to obtain N_{ij}, then calculate the weighted normalized matrix as:

$$W_{ij} = w_j \cdot N_{ij},$$

where w_j is the weight of the j-th criterion.

(3) Define the Neutrosophic Positive Ideal Solution (NPIS) and Neutrosophic Negative Ideal Solution (NNIS):

$$NPIS = \big(T^+, I^-, F^-\big), \quad NNIS = \big(T^-, I^+, F^+\big),$$

where:

$$T^+ = \max_i W_{ij}(T), \quad I^- = \min_i W_{ij}(I), \quad F^- = \min_i W_{ij}(F),$$

$$T^- = \min_i W_{ij}(T), \quad I^+ = \max_i W_{ij}(I), \quad F^+ = \max_i W_{ij}(F).$$

(4) Calculate the separation measures from NPIS and NNIS:

$$S_i^+ = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

$$S_i^- = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

(5) Compute the relative closeness of each alternative to the NPIS:

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}.$$

(6) Rank the alternatives based on the values of C_i, with higher C_i indicating better alternatives.

Definition 3.67 (HyperNeutrosophic TOPSIS)**.** Let $A = \{A_1, A_2, \ldots, A_m\}$ represent m alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ represent n criteria. In HyperNeutrosophic TOPSIS, each element of the decision matrix D is represented as a set of neutrosophic triples:

$$D_{ij} = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\},$$

where:

- $T_{ij}^k, I_{ij}^k, F_{ij}^k \in [0,1]$ are the k-th truth, indeterminacy, and falsity membership degrees for A_i under criterion C_j,
- $\mathcal{K}_{ij}$ is the index set of evaluations for D_{ij},
- Each triple satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3.$$

The HyperNeutrosophic TOPSIS algorithm consists of the following steps:

(1) Construct the Decision Matrix. The decision matrix D consists of sets of neutrosophic triples for each criterion and alternative.

(2) Aggregate Membership Degrees. Aggregate the membership degrees over $k \in \mathcal{K}_{ij}$ using an appropriate aggregation operator $\oplus$:

$$\overline{D}_{ij} = \big(T_{ij}, I_{ij}, F_{ij}\big),$$

where:

$$T_{ij} = \oplus_{k\in\mathcal{K}_{ij}} T_{ij}^k, \quad I_{ij} = \oplus_{k\in\mathcal{K}_{ij}} I_{ij}^k, \quad F_{ij} = \oplus_{k\in\mathcal{K}_{ij}} F_{ij}^k.$$

(3) Normalize the Decision Matrix. Normalize $\overline{D}_{ij}$ using:

$$N_{ij} = \left(\frac{T_{ij}}{\max_i T_{ij}}, \frac{I_{ij}}{\max_i I_{ij}}, \frac{F_{ij}}{\max_i F_{ij}}\right).$$

(4) Weight the Normalized Decision Matrix. Apply weights w_j to obtain:

$$W_{ij} = w_j \cdot N_{ij}.$$

(5) Determine the Ideal Solutions. Compute the HyperNeutrosophic Positive Ideal Solution (HPIS) and HyperNeutrosophic Negative Ideal Solution (HNIS):

$$HPIS = \big(T^+, I^-, F^-\big), \quad HNIS = \big(T^-, I^+, F^+\big),$$

where:

$$T^+ = \max_i W_{ij}(T), \quad I^- = \min_i W_{ij}(I), \quad F^- = \min_i W_{ij}(F),$$
$$T^- = \min_i W_{ij}(T), \quad I^+ = \max_i W_{ij}(I), \quad F^+ = \max_i W_{ij}(F).$$

(6) Calculate Separation Measures. Compute the separation from HPIS and HNIS:

$$S_i^+ = \sqrt{\sum_{j=1}^n \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

$$S_i^- = \sqrt{\sum_{j=1}^n \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

(7) Compute Relative Closeness. Calculate the relative closeness of each alternative to HPIS:

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}.$$

(8) Rank Alternatives. Rank alternatives based on C_i.

Definition 3.68 (n-SuperHyperNeutrosophic TOPSIS)**.** Let $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, C_2, \ldots, C_n\}$. In n-SuperHyperNeutrosophic TOPSIS, each element of the decision matrix D is recursively defined as:

$$D_{ij}^{(n)} = \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ represents the n-th power set of the neutrosophic unit cube.

The procedure extends HyperNeutrosophic TOPSIS with the following modifications:

(1) Recursive Aggregation. Aggregate the nested membership degrees recursively:

$$\overline{D}_{ij}^{(n)} = \text{Aggregate}\big(D_{ij}^{(n)}\big),$$

where the aggregation operator accounts for all nested layers.

(2) Normalize and Weight. Normalize and weight $\overline{D}_{ij}^{(n)}$ as in HyperNeutrosophic TOPSIS.

(3) Ideal Solutions. Define n-SuperHyperNeutrosophic Positive Ideal Solution (n-SHPIS) and Negative Ideal Solution (n-SHNIS) recursively.

(4) Separation Measures. Extend the separation measures to nested structures:

$$S_i^{+,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

$$S_i^{-,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

(5) Relative Closeness. Compute the relative closeness for nested structures:

$$C_i^{(n)} = \frac{S_i^{-,(n)}}{S_i^{+,(n)} + S_i^{-,(n)}}.$$

(6) Rank Alternatives. Rank based on $C_i^{(n)}$.

Theorem 3.69 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic TOPSIS possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.70 (Reduction to Neutrosophic TOPSIS)**.** *If $n = 0$, then n-SuperHyperNeutrosophic TOPSIS reduces to standard Neutrosophic TOPSIS.*

Proof. When $n = 0$, each element $D_{ij}^{(0)}$ of the decision matrix is simply a neutrosophic triple (T_{ij}, I_{ij}, F_{ij}) without any additional nesting. In this case, all steps of the method—normalization, weighting, computation of NPIS/NNIS, separation measures, relative closeness, and ranking—are identical to those of the standard Neutrosophic TOPSIS procedure.

Thus, the 0-SuperHyperNeutrosophic TOPSIS framework coincides exactly with Neutrosophic TOPSIS. □

Theorem 3.71 (Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SuperHyperNeutrosophic TOPSIS framework is strictly more complex in terms of aggregation and computation than the $(n-1)$-SuperHyperNeutrosophic TOPSIS framework.*

Proof. The n-SuperHyperNeutrosophic decision matrix $D_{ij}^{(n)}$ is defined recursively by:

$$D_{ij}^{(n)} = \mathcal{P}(D_{ij}^{(n-1)}),$$

where $\mathcal{P}$ denotes the power set operation. Each application of $\mathcal{P}$ produces an exponentially larger set of subsets, thereby increasing both the complexity of the membership structures and the effort required to aggregate them. Hence, as n increases, the amount of computational work grows significantly, making n-SuperHyperNeutrosophic TOPSIS strictly more complex than its $(n-1)$-level counterpart. □

Theorem 3.72 (Continuity of Relative Closeness)**.** *The relative closeness $C_i^{(n)}$ in n-SuperHyperNeutrosophic TOPSIS is a continuous function of the membership values and the criterion weights.*

Proof. The relative closeness for the n-level framework is given by:

$$C_i^{(n)} = \frac{S_i^{-,(n)}}{S_i^{+,(n)} + S_i^{-,(n)}},$$

where

$$S_i^{+,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

and

$$S_i^{-,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

Each component (T_{ij}, I_{ij}, F_{ij}) is drawn from a continuous domain $[0,1]^3$. Arithmetic operations, squaring, and taking square roots are all continuous mappings. Therefore, $S_i^{+,(n)}$ and $S_i^{-,(n)}$ depend continuously on the membership values and weights. As a ratio of two continuous, strictly positive expressions, $C_i^{(n)}$ itself is continuous. □

Theorem 3.73 (Convergence)**.** *The n-SuperHyperNeutrosophic TOPSIS method converges to a stable ranking of alternatives as the iterative process reaches a steady state.*

Proof. The iterative steps of the algorithm involve standardizable processes: normalization, weighting, and aggregation of nested memberships. Since each neutrosophic value (T, I, F) is bounded within $[0, 1]$, these transformations remain stable and well-defined. As iteration continues, successive recalculations of separation measures and relative closeness values approach fixed points, leading the algorithm to stabilize. Consequently, the final ranking of alternatives will not oscillate indefinitely but will converge to a stable ordering. □

Theorem 3.74 (Consistency of Ideal Solutions)**.** *The n-SuperHyperNeutrosophic Positive Ideal Solution (n-SHPIS) and Negative Ideal Solution (n-SHNIS) consistently reflect the extreme preferences defined by the weighted, normalized decision matrix* $D_{ij}^{(n)}$.

Proof. The n-SHPIS and n-SHNIS (T^+, I^-, F^-) and (T^-, I^+, F^+) are determined by extremizing the criterion values in the weighted normalized space. Because these extremizing operations directly use the aggregated values at the n-th level, they yield consistent definitions of "best" and "worst" scenarios with respect to truth, indeterminacy, and falsity dimensions. Thus, the ideal solutions accurately represent the extremal preference structure within the n-SuperHyperNeutrosophic framework, ensuring coherent and meaningful comparisons between alternatives. □

Similarly, HyperFuzzy TOPSIS, HyperPlithogenic TOPSIS, SuperHyperFuzzy TOPSIS, and SuperHyperPlithogenic TOPSIS can also be defined.

3.8. *Neutrosophic Evolution*

Darwin's Evolution explains the process of adaptation and evolution driven by natural selection [59, 125, 176, 177, 320, 430]. In contrast, Neutrosophic Evolution provides a multidimensional framework that incorporates truth, falsity, and indeterminacy to model adaptation and change [99, 286, 287, 372, 376]. This concept can also be interpreted using HyperNeutrosophic Sets or n-SuperHyperNeutrosophic Sets. The following sections outline their definitions and related formulations.

Definition 3.75 (Neutrosophic Evolution)**.** [376] Neutrosophic Evolution describes the adaptation of a being B (e.g., plant, animal, human, or system) in a given environment η over a specific timespan Δt. It considers three components for every part P of B:

$$E(P), I(P), \text{ and } N(P),$$

representing degrees of evolution, involution, and neutrality (indeterminacy), respectively.

Mathematical Representation. For a being B with parts $P_1, P_2, \ldots, P_n$, the state of each part P_i is defined as a Neutrosophic Triple:

$$S(P_i) = \big(E(P_i), I(P_i), N(P_i)\big),$$

where:

- $E(P_i) \in [0,1]$: Degree of evolution of part P_i,
- $I(P_i) \in [0,1]$: Degree of involution (degeneration) of part P_i,
- $N(P_i) \in [0,1]$: Degree of neutrality (indeterminate change) of part P_i,

and the following constraint holds:

$$0 \leq E(P_i) + I(P_i) + N(P_i) \leq 3.$$

Total Neutrosophic State of a Being. The overall state of B at time t is represented as:

$$S(B,t) = \Big(E(B,t), I(B,t), N(B,t)\Big),$$

where:

$$E(B,t) = \frac{1}{n}\sum_{i=1}^{n} E(P_i), \quad I(B,t) = \frac{1}{n}\sum_{i=1}^{n} I(P_i), \quad N(B,t) = \frac{1}{n}\sum_{i=1}^{n} N(P_i).$$

Dynamic Behavior. The evolution of B over time t is defined by the derivatives:

$$\frac{\partial E(B,t)}{\partial t}, \quad \frac{\partial I(B,t)}{\partial t}, \quad \frac{\partial N(B,t)}{\partial t}.$$

These represent the rates of change of evolution, involution, and neutrality, respectively. The dynamic system governing $S(B,t)$ is expressed as:

$$\begin{aligned}
\frac{\partial E(B,t)}{\partial t} &= f_1(E, I, N, \eta, \Delta t), \\
\frac{\partial I(B,t)}{\partial t} &= f_2(E, I, N, \eta, \Delta t), \\
\frac{\partial N(B,t)}{\partial t} &= f_3(E, I, N, \eta, \Delta t),
\end{aligned}$$

where f_1, f_2, f_3 are functions modeling the influence of environmental factors η and time Δt.

Applications. This framework can be applied to:

(1) Biological adaptation (e.g., species evolution and regression).
(2) Societal systems (e.g., cultural growth and decline).
(3) Artificial systems (e.g., machine learning model optimization and decay).

Example 3.76 (Neutrosophic Evolution in Real Life)**.** Consider the evolution of a city's traffic system B in response to the introduction of electric vehicles (EVs [129,250]) in an environment η with fluctuating adoption rates and policies over a period Δt. Each component P_i of the traffic system (e.g., road infrastructure, charging stations, traffic laws) is characterized by:

$$S(P_i) = \big(E(P_i), I(P_i), N(P_i)\big),$$

where:

- $E(P_i)$: The degree to which the component evolves (e.g., expansion of charging stations).
- $I(P_i)$: The degree of indeterminacy (e.g., uncertainty in policy implementation).
- $N(P_i)$: The degree of neutrality (e.g., components unaffected by EV adoption, like pedestrian crossings [233]).

For instance:

$$S(\text{Charging Stations}) = (0.9, 0.1, 0.0), \quad S(\text{Road Markings}) = (0.4, 0.3, 0.3), \quad S(\text{Traffic Laws}) = (0.7, 0.2, 0.1).$$

The total Neutrosophic state of the traffic system is:

$$S(B,t) = \Big(E(B,t), I(B,t), N(B,t)\Big),$$

where:

$$E(B,t) = \frac{1}{3}(0.9 + 0.4 + 0.7) = 0.67, \quad I(B,t) = \frac{1}{3}(0.1 + 0.3 + 0.2) = 0.2, \quad N(B,t) = 0.13.$$

This analysis helps city planners understand the degree of adaptation, uncertainty, and neutral impacts in the system.

Neutrosophic Evolution can be applied to a wide range of concepts. Several examples are provided below.

Example 3.77 (Examples of Neutrosophic Evolution)**.** The concept of Neutrosophic Evolution can be intuitively understood through the following real-world examples:

(1) *Biological Evolution in Urban Birds (cf. [349]):* In cities, bird species such as pigeons exhibit behavioral and physical adaptations (evolution) to urban environments, like foraging near human activity or navigating buildings. However, some traditional behaviors (e.g., nesting in natural habitats) might diminish (involution), while certain traits, such as tolerance to human presence, remain neutral (indeterminate).

(2) *Cultural Dynamics in a Society:* A society's language (cf. [163]) evolves by incorporating new slang or technical terms (evolution), while older, less-used words or traditions fade (involution). Neutral words or practices that are neither widely adopted nor completely abandoned, like regional dialects, remain in an indeterminate state.

(3) *Adaptation of Machine Learning Models:* A machine learning model trained for spam detection updates its parameters when new patterns of spam emails emerge [182] (evolution). Outdated features lose their relevance and are discarded (involution). Features that are rarely triggered by current data may stay in a neutral state until more evidence supports their inclusion or exclusion (indeterminate).

(4) *Ecosystem Response to Climate Change:* Coral reefs adapt to rising temperatures by developing heat-tolerant algae (evolution). Some species, unable to cope, face population decline or extinction (involution). Other species might exhibit no clear trend of adaptation or decline due to insufficient data or complexity of interactions (indeterminacy).

(5) *Technological Development in Renewable Energy:* Solar panel technology [417] improves with higher efficiency rates and cost reductions (evolution). Outdated manufacturing methods become obsolete (involution). Experimental methods, such as new types of photovoltaic cells, remain under investigation, representing an indeterminate state until proven effective or ineffective.

Definition 3.78 (HyperNeutrosophic Evolution)**.** HyperNeutrosophic Evolution extends Neutrosophic Evolution by incorporating multiple evaluations for each component (evolution, involution, and neutrality) of a being B. For each part P of B, the state is represented as:

$$H(P) = \big\{(E_k(P), I_k(P), N_k(P)) \mid k \in \mathcal{K}_P\big\},$$

where:

- $E_k(P), I_k(P), N_k(P) \in [0,1]$ are the k-th evaluations of evolution, involution, and neutrality for P,
- $\mathcal{K}_P$ is the index set of evaluations for P,
- Each triple satisfies:

$$0 \leq E_k(P) + I_k(P) + N_k(P) \leq 3.$$

Total HyperNeutrosophic State of a Being. The overall state of B at time t is given by:

$$H(B,t) = \Big(H_E(B,t), H_I(B,t), H_N(B,t)\Big),$$

where:

$$H_E(B,t) = \frac{1}{n}\sum_{i=1}^{n}\Big\{ \oplus_{k\in\mathcal{K}_{P_i}} E_k(P_i)\Big\}, \quad H_I(B,t) = \frac{1}{n}\sum_{i=1}^{n}\Big\{ \oplus_{k\in\mathcal{K}_{P_i}} I_k(P_i)\Big\},$$

$$H_N(B,t) = \frac{1}{n}\sum_{i=1}^{n}\Big\{ \oplus_{k\in\mathcal{K}_{P_i}} N_k(P_i)\Big\}.$$

Here, $\oplus$ is an aggregation operator over multiple evaluations.

Dynamic Behavior. The rates of change are defined as:

$$\frac{\partial H_E(B,t)}{\partial t}, \quad \frac{\partial H_I(B,t)}{\partial t}, \quad \frac{\partial H_N(B,t)}{\partial t}.$$

These are governed by:

$$\frac{\partial H_E(B,t)}{\partial t} = f_1(H_E, H_I, H_N, \eta, \Delta t),$$
$$\frac{\partial H_I(B,t)}{\partial t} = f_2(H_E, H_I, H_N, \eta, \Delta t),$$
$$\frac{\partial H_N(B,t)}{\partial t} = f_3(H_E, H_I, H_N, \eta, \Delta t).$$

Example 3.79 (HyperNeutrosophic Evolution in Real Life)**.** Consider the adaptation of an international healthcare system B during a global pandemic (cf. [266,455]). Each component P_i of the system (e.g., hospital infrastructure, vaccination programs [77], public health campaigns [431]) is evaluated across multiple regions or agencies $k \in \mathcal{K}_{P_i}$, leading to:

$$H(P_i) = \big\{(E_k(P_i), I_k(P_i), N_k(P_i)) \mid k \in \mathcal{K}_{P_i}\big\}.$$

For example:

$$H(\text{Hospitals}) = \big\{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1)\big\}, \quad H(\text{Vaccination Program}) = \big\{(0.9, 0.05, 0.05), (0.85, 0.1, 0.05)\big\}.$$

The total HyperNeutrosophic state of the healthcare system is:

$$H(B,t) = \Big(H_E(B,t), H_I(B,t), H_N(B,t)\Big),$$

where:

$$H_E(B,t) = \frac{1}{2}\big\{(0.8 \oplus 0.7), (0.9 \oplus 0.85)\big\} = \big\{0.75, 0.875\big\},$$

and similarly for $H_I(B,t)$ and $H_N(B,t)$. Here, $\oplus$ represents an aggregation operator, such as averaging or maximum selection.

This analysis allows stakeholders to understand how different components of the healthcare system adapt under varying regional constraints, uncertainties, and neutral factors, providing a comprehensive framework for policy-making and resource allocation.

Definition 3.80 (n-SuperHyperNeutrosophic Evolution)**.** n-SuperHyperNeutrosophic Evolution is a hierarchical generalization of HyperNeutrosophic Evolution. For each part P of B, the state is recursively defined as:

$$H^{(n)}(P) = \mathcal{P}^n([0,1]^3),$$

where:

- $\mathcal{P}^n([0,1]^3)$ is the n-th nested power set of $[0,1]^3$,
- $H^{(0)}(P) = (E(P), I(P), N(P))$ as in Neutrosophic Evolution,
- $H^{(1)}(P) = H(P)$ as in HyperNeutrosophic Evolution.

Total n-SuperHyperNeutrosophic State of a Being. The total state of B is:

$$H^{(n)}(B,t) = \Big(H_E^{(n)}(B,t), H_I^{(n)}(B,t), H_N^{(n)}(B,t)\Big),$$

where:

$$H_E^{(n)}(B,t) = \frac{1}{n}\sum_{i=1}^{n} \text{Aggregate}\big(H_E^{(n-1)}(P_i)\big),$$

and similar definitions for $H_I^{(n)}(B,t)$ and $H_N^{(n)}(B,t)$.

Dynamic Behavior. The dynamic system for n-SuperHyperNeutrosophic Evolution is:

$$\frac{\partial H_E^{(n)}(B,t)}{\partial t} = f_1^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_I^{(n)}(B,t)}{\partial t} = f_2^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_N^{(n)}(B,t)}{\partial t} = f_3^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t).$$

Theorem 3.81 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Evolution possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.82 (Reduction to Neutrosophic Evolution)**.** *For $n = 0$, n-SuperHyperNeutrosophic Evolution reduces to standard Neutrosophic Evolution.*

Proof. By definition, 0-SuperHyperNeutrosophic Evolution means no power set operation is applied. Thus, for each part P of the being B:

$$H^{(0)}(P) = (E(P), I(P), N(P)),$$

which corresponds exactly to the Neutrosophic Evolution definition. The average measures $E(B,t), I(B,t), N(B,t)$ and their dynamics are then identical to those in the original Neutrosophic Evolution framework. Hence, 0-SHN Evolution coincides with Neutrosophic Evolution. □

Theorem 3.83 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, n-SuperHyperNeutrosophic Evolution is strictly more complex than $(n-1)$-SuperHyperNeutrosophic Evolution in terms of state representation.*

Proof. The state representation for each part P at the n-th level involves:

$$H^{(n)}(P) = \mathcal{P}(H^{(n-1)}(P)),$$

where $\mathcal{P}$ is the power set operation. Since $\mathcal{P}$ applied to a non-empty set yields a strictly larger and more complex family of subsets, the complexity of the state representation grows as n increases. Each level n introduces an exponentially richer structure of nested sets of triples (E, I, N). Hence, the complexity of $H^{(n)}(P)$ strictly exceeds that of $H^{(n-1)}(P)$. □

Theorem 3.84 (Existence and Uniqueness of Solutions)**.** *Assume the functions $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ governing the n-SuperHyperNeutrosophic Evolution dynamics are continuous and satisfy Lipschitz conditions in the variables $(H_E^{(n)}, H_I^{(n)}, H_N^{(n)})$. Then, for given initial conditions $(H_E^{(n)}(B,0), H_I^{(n)}(B,0), H_N^{(n)}(B,0))$, there exists a unique local solution for the dynamical system:*

$$\frac{\partial H_E^{(n)}(B,t)}{\partial t} = f_1^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_I^{(n)}(B,t)}{\partial t} = f_2^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_N^{(n)}(B,t)}{\partial t} = f_3^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t).$$

Proof. The system of partial differential equations in time can be reduced to an equivalent system of ordinary differential equations if we consider spatial variables fixed or if the model is spatially lumped. Under standard ODE existence and uniqueness theorems (such as Picard–Lindelöf), continuity and Lipschitz conditions on $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ ensure a unique local solution exists through the given initial state. Thus, standard ODE theory applies, guaranteeing existence and uniqueness. □

Theorem 3.85 (Continuity under Parameter Perturbations)**.** *If the functions $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ and the aggregation operators used at level n are continuous, then the solution $(H_E^{(n)}(B,t), H_I^{(n)}(B,t), H_N^{(n)}(B,t))$ depends continuously on environmental parameters η and time scale Δt.*

Proof. Since the state $(H_E^{(n)}, H_I^{(n)}, H_N^{(n)})$ at each time t is obtained by integrating a continuous system of differential equations with continuous initial conditions and parameters, small perturbations in η or Δt yield correspondingly small changes in the solutions. Continuity of the aggregation operators ensures that nested evaluations do not introduce discontinuities. Therefore, the entire n-SHN Evolution framework is stable under small parameter perturbations. □

Theorem 3.86 (Boundedness of the State Space)**.** *For each part P and each triple $(E, I, N) \in [0,1]^3$, the nested structures remain within $\mathcal{P}^n([0,1]^3)$. Therefore, the state space for n-SHN Evolution is bounded.*

Proof. By construction, each neutrosophic triple (E, I, N) satisfies $E, I, N \in [0, 1]$. The power set operation preserves boundedness as it does not introduce elements outside the original set. Hence, for all n, the state $H^{(n)}(P) \subseteq \mathcal{P}^n([0, 1]^3)$ is confined to a bounded domain. Consequently, $H^{(n)}(B, t)$ remains within a bounded state space for all t. □

Similarly, HyperFuzzy Evolution, HyperPlithogenic Evolution, SuperHyperFuzzy Evolution, and SuperHyperPlithogenic Evolution can also be defined.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[2] Hareem Abbasi, Usman Shahzad, Walid Emam, Muhammad Hanif, Nasir Ali, and Mubeen Mukhtar. Calibrated empirical neutrosophic cumulative distribution function estimation for both symmetric and asymmetric data. *Symmetry*, 16:633, 2024.

[3] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[4] Mohamed Abdel-Basset, Mai Mohamed, and Florentin Smarandache. A hybrid neutrosophic group anp-topsis framework for supplier selection problems. *Symmetry*, 10:226, 2018.

[5] Mohamed Abdel-Basset and Rehab Mohamed. A novel plithogenic topsis-critic model for sustainable supply chain risk management. *Journal of Cleaner production*, 247:119586, 2020.

[6] Ahmed Abdelhafeez, Hoda K Mohamed, Ali Maher, and Ahmed Abdelmonem. A neutrosophic based c-means approach for improving breast cancer clustering performance. *Neutrosophic Sets and Systems*, 53:317–330, 2023.

[7] Josh Achiam, Steven Adler, Sandhini Agarwal, Lama Ahmad, Ilge Akkaya, Florencia Leoni Aleman, Diogo Almeida, Janko Altenschmidt, Sam Altman, Shyamal Anadkat, et al. Gpt-4 technical report. *arXiv preprint arXiv:2303.08774*, 2023.

[8] J K Adarsh, V. T. Sreedevi, and Deepa Thangavelusamy. Product review system with bert for sentiment analysis and implementation of administrative privileges on node-red. *IEEE Access*, 11:65968–65976, 2023.

[9] Nehmat Ahmed and Osama T. Pirbal. Plithogenic crisp hypersoft topology. *European Journal of Pure and Applied Mathematics*, 2024.

[10] Yaman Akbulut, Abdulkadir Şengür, Yanhui Guo, and Kemal Polat. Kncm: Kernel neutrosophic c-means clustering. *Applied Soft Computing*, 52:714–724, 2017.

[11] Charles O Akinyokun, Okure U Obot, Faith-Michael E Uzoka, and John J Andy. A neuro-fuzzy decision support system for the diagnosis of heart failure. In *Medical and care compunetics 6*, pages 231–244. IOS Press, 2010.

[12] M. Akram and K. H. Dar. On n-graphs. *Southeast Asian Bulletin of Mathematics*, 38:35–49, 2014.

[13] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[14] Muhammad Akram. Interval-valued fuzzy line graphs. *Neural Computing and Applications*, 21:145–150, 2012.

[15] Muhammad Akram, Arooj Adeel, and José Carlos Rodriguez Alcantud. Group decision-making methods based on hesitant n-soft sets. *Expert Syst. Appl.*, 115:95–105, 2019.

[16] Muhammad Akram, Arooj Adeel, and José Carlos Rodriguez Alcantud. Hesitant fuzzy n-soft sets: A new model with applications in decision-making. *J. Intell. Fuzzy Syst.*, 36:6113–6127, 2019.

[17] Muhammad Akram and Noura Omair Alshehri. Tempered interval-valued fuzzy hypergraphs. *University politehnica of bucharest scientific bulletin-series a-applied mathematics and physics*, 77(1):39–48, 2015.

[18] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.

[19] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[20] Muhammad Akram, Feng Feng, Shahzad Sarwar, and Youne Bae Jun. Certain types of vague graphs. *University Politehnica of Bucharest Scientific Bulletin Series A*, 76(1):141–154, 2014.

[21] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[22] Muhammad Akram, MG Karunambigai, K Palanivel, and S Sivasankar. Balanced bipolar fuzzy graphs. *Journal of advanced research in pure mathematics*, 6(4):58–71, 2014.

[23] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, 31(56):425–438, 2013.

[24] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[25] Muhammad Akram, Sundas Shahzadi, Areen Rasool, and Musavarah Sarwar. Decision-making methods based on fuzzy soft competition hypergraphs. *Complex & Intelligent Systems*, 8(3):2325–2348, 2022.

[26] Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.

[27] Muhammad Akram, Shumaiza, and José Carlos Rodríguez Alcantud. Multi-criteria decision making methods with bipolar fuzzy sets. *Forum for Interdisciplinary Mathematics*, 2023.

[28] Muhammad Akram, Shumaiza, and Florentin Smarandache. Decision-making with bipolar neutrosophic topsis and bipolar neutrosophic electre-i. *Axioms*, 7:33, 2018.

[29] Yousef Al-Qudah and Nasruddin Hassan. Operations on complex multi-fuzzy sets. *J. Intell. Fuzzy Syst.*, 33:1527–1540, 2017.

[30] Tareq M. Al-shami, Ljubisa D. R. Kocinac, and Baravan A. Asaad. Sum of soft topological spaces. *Mathematics*, 2020.

[31] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. New concepts in partner multineutrosophic topological space. *International Journal of Neutrosophic Science*, (3):172–72, 2024.

[32] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[33] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[34] Moslem Alimohammadlou and Zahra Khoshsepehr. The role of society 5.0 in achieving sustainable development: a spherical fuzzy set approach. *Environmental Science and Pollution Research*, 30:47630–47654, 2023.

[35] Abd Ulazeez M. Alkouri, Mo'men Bany Amer, and Ibrahim Jawarneh. Complex bipolar multi-fuzzy sets. *Uncertainty*, 2024.

[36] James Allen. *Natural language understanding*. Benjamin-Cummings Publishing Co., Inc., 1988.

[37] Marcos Antônio Alves, Petrônio Cândido de Lima e Silva, Carlos Alberto Severiano Junior, Gustavo Linhares Vieira, Frederico Gadelha Guimarães, and Hossein Javedani Sadaei. An extension of nonstationary fuzzy sets to heteroskedastic fuzzy time series. In *ESANN*, 2018.

[38] Ion Androutsopoulos, Graeme D Ritchie, and Peter Thanisch. Natural language interfaces to databases–an introduction. *Natural language engineering*, 1(1):29–81, 1995.

[39] Kenneth J. Arrow. Social choice and individual values. 1951.

[40] Baravan A Asaad, Sagvan Y Musa, and Zanyar A Ameen. Fuzzy bipolar hypersoft sets: A novel approach for decision-making applications. *Mathematical and Computational Applications*, 29(4):50, 2024.

[41] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani, and Tariq Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.*, 36:2829–2844, 2019.

[42] Muhammad Aslam. Introducing grubbs's test for detecting outliers under neutrosophic statistics - an application to medical data. *Journal of King Saud University - Science*, 2020.

[43] Muhammad Aslam. On detecting outliers in complex data using dixon's test under neutrosophic statistics. *Journal of King Saud University - Science*, 2020.

[44] Muhammad Aslam and Mohammed Albassam. Presenting post hoc multiple comparison tests under neutrosophic statistics. *Journal of King Saud University - Science*, 2020.

[45] Muhammad Aslam and Osama H. Arif. Testing of grouped product for the weibull distribution using neutrosophic statistics. *Symmetry*, 10:403, 2018.

[46] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[47] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[48] Krassimir T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[49] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[50] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. *Interval-Valued Intuitionistic Fuzzy Sets*, 2019.

[51] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[52] Krassimir T. Atanassov. New topological operator over intuitionistic fuzzy sets. 2023.

[53] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[54] Malik Shahzad Kaleem Awan and Mian Muhammad Awais. Predicting weather events using fuzzy rule based system. *Applied Soft Computing*, 11(1):56–63, 2011.

[55] Jawid Ahmad Baktash and Mursal Dawodi. Gpt-4: A review on advancements and opportunities in natural language processing. *arXiv preprint arXiv:2305.03195*, 2023.

[56] Jonathan D Barsky and Richard Labagh. A strategy for customer satisfaction. *Cornell Hotel and Restaurant Administration Quarterly*, 33(5):32–40, 1992.

[57] Rajamanickam Baskar, Kuo Ann Lee, Richard Yeo, and Kheng-Wei Yeoh. Cancer and radiation therapy: current advances and future directions. *International journal of medical sciences*, 9(3):193, 2012.

[58] Sadi Bayramov and Cigdem G. Aras. A new approach to separability and compactness in soft topological spaces. 2018.

[59] Michael J Behe. *Darwin's black box: The biochemical challenge to evolution*. Simon and Schuster, 1996.

[60] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[61] Anol Bhattacherjee. Understanding information systems continuance: An expectation-confirmation model. *MIS Q.*, 25:351–370, 2001.

[62] Monoranjan Bhowmik and Madhumangal Pal. *Intuitionistic neutrosophic set*. Infinite Study, 2009.

[63] Monoranjan Bhowmik and Madhumangal Pal. Intuitionistic neutrosophic set relations and some of its properties. *Journal of Information and Computing Science*, 5(3):183–192, 2010.

[64] S Bhuvaneshwari and C Antony Crispin Sweety. Neutrosophic spherical sets in mcdm. *Neutrosophic Sets and Systems*, 68:136–153, 2024.

[65] Steven Bird, Ewan Klein, and Edward Loper. *Natural language processing with Python: analyzing text with the natural language toolkit*. " O'Reilly Media, Inc.", 2009.

[66] Anamika Bisane, Shivanand Chandravanshi, Palash Manoj Thakre, Purab Kesharwani, and Atiya Khan. A comprehensive product review system for improved customer satisfaction. *2023 2nd International Conference on Paradigm Shifts in Communications Embedded Systems, Machine Learning and Signal Processing (PCEMS)*, pages 1–6, 2023.

[67] Isabelle Bloch. Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. *Inf. Sci.*, 181:2002–2015, 2011.

[68] Isabelle Bloch. Mathematical morphology on bipolar fuzzy sets: general algebraic framework. *Int. J. Approx. Reason.*, 53:1031–1060, 2012.

[69] Borzoo Bonakdarpour and Sarai Sheinvald. Automata for hyperlanguages. *arXiv preprint arXiv:2002.09877*, 2020.

[70] Borzoo Bonakdarpour and Sarai Sheinvald. Finite-word hyperlanguages. *Information and Computation*, 295:104944, 2023.

[71] Gourangajit Borah and Palash Dutta. Fuzzy risk analysis in crop selection using information measures on quadripartitioned single-valued neutrosophic sets. *Expert Syst. Appl.*, 255:124750, 2024.

[72] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[73] Samuel R Bowman, Gabor Angeli, Christopher Potts, and Christopher D Manning. A large annotated corpus for learning natural language inference. *arXiv preprint arXiv:1508.05326*, 2015.

[74] George EP Box, J Stuart Hunter, William G Hunter, et al. Statistics for experimenters. In *Wiley series in probability and statistics*. Wiley Hoboken, NJ, 2005.

[75] George EP Box, William H Hunter, Stuart Hunter, et al. *Statistics for experimenters*, volume 664. John Wiley and sons New York, 1978.

[76] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[77] Marc Brisson and W. John Edmunds. Economic evaluation of vaccination programs: The impact of herd-immunity. *Medical Decision Making*, 23:76 – 82, 2003.

[78] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[79] Said Broumi. A single valued neutrosophic inventory model with neutrosophic random variable. *International Journal of Neutrosophic Science*, 2020.

[80] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[81] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[82] Richard Hubert Bruck. *A survey of binary systems*, volume 20. Springer, 1971.

[83] James J. Buckley. Fuzzy statistics: hypothesis testing. *Soft Computing*, 9:512–518, 2005.

[84] James J. Buckley. Fuzzy probability and statistics. 2006.

[85] M Burgin and A Krinik. Introduction to conditional hyperprobabilities. *Integration*, 2(3):285, 2011.

[86] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[87] Walter Alexandre Carnielli and Marcelo Esteban Coniglio. Paraconsistent set theory by predicating on consistency. *J. Log. Comput.*, 26:97–116, 2016.

[88] Susumu Cato. Quasi-decisiveness, quasi-ultrafilter, and social quasi-orderings. *Social Choice and Welfare*, 41(1):169–202, 2013.

[89] Alain Chateauneuf. Continuous representation of a preference relation on a connected topological space. *Journal of Mathematical Economics*, 16:139–146, 1987.

[90] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 30:2475–2485, 2015.

[91] AK Chaudhuri and P Das. Fuzzy connected sets in fuzzy topological spaces. *Fuzzy Sets and Systems*, 49(2):223–229, 1992.

[92] Chen-Tung Chen. Extensions of the topsis for group decision-making under fuzzy environment. *Fuzzy Sets Syst.*, 114:1–9, 2000.

[93] Jiqian Chen, Jun Ye, and Shigui Du. Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry*, 9:208, 2017.

[94] Qian Chen, Xiaodan Zhu, Zhenhua Ling, Si Wei, Hui Jiang, and Diana Inkpen. Enhanced lstm for natural language inference. *arXiv preprint arXiv:1609.06038*, 2016.

[95] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[96] Wei Chen, Wei Chen, Hamid Reza Pourghasemi, M. Panahi, Aiding Kornejady, Jiale Wang, Xiaoshen Xie, and Shubo Cao. Spatial prediction of landslide susceptibility using an adaptive neuro-fuzzy inference system combined with frequency ratio, generalized additive model, and support vector machine techniques. *Geomorphology*, 297:69–85, 2017.

[97] Daizhan Cheng, Xiaoming Hu, and Tielong Shen. Topological space.

[98] KR1442 Chowdhary and KR Chowdhary. Natural language processing. *Fundamentals of artificial intelligence*, pages 603–649, 2020.

[99] Victor Christianto. Remark on neutrosophic genetics: A generalized way towards five schemes of mutation and transmutation. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 42, 2021.

[100] Gilbert A Churchill Jr and Carol Surprenant. An investigation into the determinants of customer satisfaction. *Journal of marketing research*, 19(4):491–504, 1982.

[101] Ronan Collobert, Jason Weston, Léon Bottou, Michael Karlen, Koray Kavukcuoglu, and Pavel Kuksa. Natural language processing (almost) from scratch. *Journal of machine learning research*, 12:2493–2537, 2011.

[102] Alexis Conneau, Douwe Kiela, Holger Schwenk, Loïc Barrault, and Antoine Bordes. Supervised learning of universal sentence representations from natural language inference data. *arXiv preprint arXiv:1705.02364*, 2017.

[103] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[104] Bui Cong Cuong and Pham Van Hai. Some fuzzy logic operators for picture fuzzy sets. *2015 Seventh International Conference on Knowledge and Systems Engineering (KSE)*, pages 132–137, 2015.

[105] Rakhal Das. Spatial fuzzy topological space. *Proyecciones (Antofagasta)*, 2022.

[106] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[107] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[108] José R González de Mendívil. A generalization of myhill–nerode theorem for fuzzy languages. *Fuzzy Sets and Systems*, 301:103–115, 2016.

[109] José R González de Mendívil and Jose R Garitagoitia. Fuzzy languages with infinite range accepted by fuzzy automata: Pumping lemma and determinization procedure. *Fuzzy Sets and Systems*, 249:1–26, 2014.

[110] Aurore Delaigle and Peter Hall. Defining probability density for a distribution of random functions. *arXiv: Statistics Theory*, 2010.

[111] Irfan Deli. Refined neutrosophic sets and refined neutrosophic soft sets: theory and applications. In *Handbook of research on generalized and hybrid set structures and applications for soft computing*, pages 321–343. IGI Global, 2016.

[112] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[113] Irfan Deli, S Yusuf, Florentin Smarandache, and Mumtaz Ali. Interval valued bipolar neutrosophic sets and their application in pattern recognition. In *IEEE world congress on computational intelligence*, 2016.

[114] Nandini Dendukuri and Lawrence Joseph. Bayesian approaches to modeling the conditional dependence between multiple diagnostic tests. *Biometrics*, 57, 2001.

[115] G. Devika and Sudhaman Parthasarathy. Fuzzy statistics-based affinity propagation technique for clustering in satellite cloud image. *European Journal of Remote Sensing*, 51:754 – 764, 2018.

[116] Vincent T DeVita Jr and Edward Chu. A history of cancer chemotherapy. *Cancer research*, 68(21):8643–8653, 2008.

[117] Jacob Devlin. Bert: Pre-training of deep bidirectional transformers for language understanding. *arXiv preprint arXiv:1810.04805*, 2018.

[118] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[119] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[120] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[121] Sayipjamal Dulat, Sayipjamal Dulat, Tie-Jiun Hou, Jun Gao, Marco Guzzi, Joey Huston, Pavel M. Nadolsky, Jon Pumplin, Carl R. Schmidt, Daniel Robert Stump, and C.-P. Yuan. New parton distribution functions from a global analysis of quantum chromodynamics. *Physical Review D*, 93:033006, 2015.

[122] Ward Edwards. The theory of decision making. *Psychological bulletin*, 51(4):380, 1954.

[123] Kathleen M Eisenhardt and Mark J Zbaracki. Strategic decision making. *Strategic management journal*, 13(S2):17–37, 1992.

[124] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[125] Alvar Ellegård. *Darwin and the general reader: the reception of Darwin's theory of evolution in the British periodical press, 1859-1872*. University of Chicago Press, 1990.

[126] Ilijas Farah. Approximate homomorphisms. *Combinatorica*, 18:335–348, 1998.

[127] Nabilla Farah, Raissa Maharani, Novandra Rhezza Pratama, and M. Dachyar. E-commerce promotional products selection using swara and topsis. *International Journal of Innovative Science and Research Technology (IJISRT)*, 2024.

[128] Shuang Feng and C.L. Philip Chen. Fuzzy broad learning system: A novel neuro-fuzzy model for regression and classification. *IEEE Transactions on Cybernetics*, 50:414–424, 2020.

[129] Xuning Feng, Minggao Ouyang, Xiang Liu, Languang Lu, Yong Xia, and Xiangming He. Thermal runaway mechanism of lithium ion battery for electric vehicles: A review. *Energy Storage Materials*, 10:246–267, 2018.

[130] Abdolvahhab Fetanat, Mohsen Tayebi, and Mehran Moteraghi. Technology evaluation for biogas production from animal waste in circular carbon economy: A complex spherical fuzzy set-based decision-making framework. *Bioresource Technology Reports*, 2023.

[131] Aldo Figallo-Orellano and Juan Sebastian Slagter. Models for da costa's paraconsistent set theory. 2020.

[132] Hadar Frenkel and Sarai Sheinvald. Realizable and context-free hyperlanguages. *arXiv preprint arXiv:2209.10306*, 2022.

[133] Kevin Fuchs. Exploring the opportunities and challenges of nlp models in higher education: is chat gpt a blessing or a curse? In *Frontiers in Education*, volume 8, page 1166682. Frontiers Media SA, 2023.

[134] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 437.

[135] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. 2024.

[136] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[137] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[138] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. 2024.

[139] Takaaki Fujita. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 86, 2024.

[140] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[141] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[142] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[143] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). 2024.

[144] Takaaki Fujita. Theoretical interpretations of large uncertain and hyper language models: Advancing natural uncertain and hyper language processing. 2024.

[145] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). 2024.

[146] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[147] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[148] Takaaki Fujita and Smarandache Florentin. Some graph parameters for superhypertree-width and neutrosophictree-width. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[149] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[150] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[151] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[152] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[153] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[154] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[155] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[156] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[157] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[158] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434, 2023.

[159] Abdul Haseeb Ganie, Surender Singh, and P. K. Bhatia. Some new correlation coefficients of picture fuzzy sets with applications. *Neural Computing and Applications*, 32:12609 – 12625, 2020.

[160] Rui Gao and Jianrong Wu. Filter with its applications in fuzzy soft topological spaces. *AIMS Mathematics*, 6(3):2359–2368, 2021.

[161] Jonathan M Garibaldi, Marcin Jaroszewski, and Salang Musikasuwan. New concepts related to non-stationary fuzzy sets. In *2007 IEEE International Fuzzy Systems Conference*, pages 1–6. IEEE, 2007.

[162] Audrey P. Gasch and Michael B. Eisen. Exploring the conditional coregulation of yeast gene expression through fuzzy k-means clustering. *Genome Biology*, 3:research0059.1 – research0059.22, 2002.

[163] James Paul Gee. Social linguistics and literacies: Ideology in discourse. 1996.

[164] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[165] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[166] Gerd Gigerenzer and Wolfgang Gaissmaier. Heuristic decision making. *Annual review of psychology*, 62(1):451–482, 2011.

[167] Allen S Ginsberg and Fred L Offensend. An application of decision theory to a medical diagnosis-treatment problem. *IEEE Transactions on Systems Science and Cybernetics*, 4(3):355–362, 1968.

[168] Yoav Goldberg. *Neural network methods in natural language processing*. Morgan & Claypool Publishers, 2017.

[169] S Gomathi, M Karpagadevi, S Krishnaprakash, A Revathy, and Said Broumi. Cubic spherical neutrosophic sets for advanced decision-making. *Neutrosophic Sets and Systems*, 73(1):24, 2024.

[170] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[171] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[172] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[173] Georg Gottlob and Reinhard Pichler. Hypergraphs in model checking: Acyclicity and hypertree-width versus clique-width. *SIAM Journal on Computing*, 33(2):351–378, 2004.

[174] Carlos Granados. New notions on neutrosophic random variables. 2021.

[175] Carlos Granados and José Rubén Sanabria. On independence neutrosophic random variables. 2021.

[176] Peter R Grant. *Ecology and Evolution of Darwin's Finches (Princeton Science Library Edition)*. Princeton University Press, 2017.

[177] Peter R Grant and B Rosemary Grant. *40 Years of Evolution: Darwin's Finches on Daphne Major Island, New Edition*. Princeton University Press, 2024.

[178] Khara D. Grieger, Igor Linkov, Steffen Foss Hansen, and Anders Baun. Environmental risk analysis for nanomaterials: Review and evaluation of frameworks. *Nanotoxicology*, 6:196 – 212, 2012.

[179] Abhishek Guleria and Rakesh Kumar Bajaj. Eigen spherical fuzzy set and its application in decision making problem. *Scientia Iranica*, 2019.

[180] Fatma Kutlu Gündogdu and Cengiz Kahraman. A novel vikor method using spherical fuzzy sets and its application to warehouse site selection. *J. Intell. Fuzzy Syst.*, 37:1197–1211, 2019.

[181] Fatma Kutlu Gündogdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *J. Intell. Fuzzy Syst.*, 36:337–352, 2019.

[182] Yanhui Guo, Zelal Mustafaoglu, and Deepika Koundal. Spam detection using bidirectional transformers and machine learning classifier algorithms. 2023.

[183] Yanhui Guo and Abdulkadir Sengur. Ncm: Neutrosophic c-means clustering algorithm. *Pattern Recognition*, 48(8):2710–2724, 2015.

[184] Himanshu Gupta and Mukesh Kumar Barua. Supplier selection among smes on the basis of their green innovation ability using bwm and fuzzy topsis. *Journal of Cleaner Production*, 152:242–258, 2017.

[185] RK Gupta, Rama Mehta, Vijaya Agarwala, Bhanu Pant, and PP Sinha. Ductility prediction of ti aluminide intermetallics through neuro-fuzzy set approach. *Transactions of the Indian Institute of Metals*, 63:833–839, 2010.

[186] Mostafa Hajiaghaei-Keshteli, Zeynep Cenk, Babek Erdebilli, Yavuz Selim Özdemir, and Fatemeh Gholian-Jouybari. Pythagorean fuzzy topsis method for green supplier selection in the food industry. *Expert Syst. Appl.*, 224:120036, 2023.

[187] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[188] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[189] Walid Hariri. Unlocking the potential of chatgpt: A comprehensive exploration of its applications, advantages, limitations, and future directions in natural language processing. *arXiv preprint arXiv:2304.02017*, 2023.

[190] Michael A Harrison. *Introduction to formal language theory*. Addison-Wesley Longman Publishing Co., Inc., 1978.

[191] John A. Hartigan and M. Anthony. Wong. A k-means clustering algorithm. 1979.

[192] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[193] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[194] Gary G Hendrix, Earl D Sacerdoti, Daniel Sagalowicz, and Jonathan Slocum. Developing a natural language interface to complex data. *ACM Transactions on Database Systems (TODS)*, 3(2):105–147, 1978.

[195] Doeko Homan. Very basic set theory. 2023.

[196] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[197] Haoyuan Hong, M. Panahi, Ataollah Shirzadi, Tianwu Ma, Junzhi Liu, A-Xing Zhu, Wei Chen, Ioannis Kougias, and Nerantzis Kazakis. Flood susceptibility assessment in hengfeng area coupling adaptive neuro-fuzzy inference system with genetic algorithm and differential evolution. *The Science of the total environment*, 621:1124–1141, 2018.

[198] John E Hopcroft and Jeffrey D Ullman. *Formal languages and their relation to automata*. Addison-Wesley Longman Publishing Co., Inc., 1969.

[199] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12:57, 2023.

[200] Han-Chen Huang and Xiao-Jun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 24:213–228, 2016.

[201] Han-Chen Huang and Xiaojun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):213–227, 2016.

[202] S Satham Hussain, Isnaini Rosyida, Hossein Rashmanlou, and Farshid Mofidnakhaei. Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis. *Computational and Applied Mathematics*, 40(4):121, 2021.

[203] Abiodun Motunrayo Ikotun, Ezugwu E. Absalom, Laith Mohammad Abualigah, Belal Abuhaija, and Heming Jia. K-means clustering algorithms: A comprehensive review, variants analysis, and advances in the era of big data. *Inf. Sci.*, 622:178–210, 2022.

[204] Maria Infusino. Topological vector spaces. 2018.

[205] Parthasarathy Iswarya and Dr. K. Bageerathi. On neutrosophic semi-open sets in neutrosophic topological spaces. *International Journal of Mathematics Trends and Technology*, 37:214–223, 2016.

[206] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[207] Sirus Jahanpanah and Roohallah Daneshpayeh. On derived superhyper be-algebras. *Neutrosophic Sets and Systems*, 57(1):21, 2023.

[208] Sirus Jahanpanah and Roohallah Daneshpayeh. An outspread on valued logic superhyperalgebras. *Facta Universitatis, Series: Mathematics and Informatics*, pages 427–437, 2024.

[209] Anil K. Jain. Data clustering: 50 years beyond k-means. *Pattern Recognit. Lett.*, 31:651–666, 2008.

[210] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[211] Ziwei Ji, Nayeon Lee, Rita Frieske, Tiezheng Yu, Dan Su, Yan Xu, Etsuko Ishii, Ye Jin Bang, Andrea Madotto, and Pascale Fung. Survey of hallucination in natural language generation. *ACM Computing Surveys*, 55(12):1–38, 2023.

[212] Wenxiang Jiao, Wenxuan Wang, Jen tse Huang, Xing Wang, and Zhaopeng Tu. Is chatgpt a good translator? yes with gpt-4 as the engine. 2023.

[213] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[214] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[215] Young Bae Jun, Min Su Kang, and Seok Zun Song. Several types of bipolar fuzzy hyper bck-ideals in hyper bck-algebras. 2012.

[216] Young Bae Jun, Seon Jeong Kim, and Seok Zun Song. Hyper permeable values and energetic sets in bck/bci-algebras. 2020.

[217] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[218] Muhammad Junaid, Ye Xue, Muzzammil Wasim Syed, Ji Zu Li, and Muhammad Ziaullah. A neutrosophic ahp and topsis framework for supply chain risk assessment in automotive industry of pakistan. *Sustainability*, 2019.

[219] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[220] Ilanthenral Kandasamy, WB Vasantha, Jagan M Obbineni, and Florentin Smarandache. Sentiment analysis of tweets using refined neutrosophic sets. *Computers in Industry*, 115:103180, 2020.

[221] Abraham Kandel and William J. Byatt. Fuzzy sets, fuzzy algebra, and fuzzy statistics. *Proceedings of the IEEE*, 66:1619–1639, 1978.

[222] Tapas Kanungo, David M. Mount, Nathan S. Netanyahu, Christine D. Piatko, Ruth Silverman, and Angela Y. Wu. An efficient k-means clustering algorithm: Analysis and implementation. *IEEE Trans. Pattern Anal. Mach. Intell.*, 24:881–892, 2002.

[223] Faruk Karaaslan. *Neutrosophic soft sets with applications in decision making*. Infinite Study, 2014.

[224] Abdullah Kargın, Florentin Smarandache, and Memet Şahin. *New Type Hyper Groups, New Type Super-Hyper Groups and Neutro-New Type SuperHyper Groups*. Infinite Study, 2023.

[225] M Kaviyarasu, Muhammad Aslam, Farkhanda Afzal, Maha Mohammed Saeed, Arif Mehmood, and Saeed Gul. The connectivity indices concept of neutrosophic graph and their application of computer network, highway system and transport network flow. *Scientific Reports*, 14(1):4891, 2024.

[226] John L. Kelley, Isaac Namioka, W. F. Donoghue, Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith. Linear topological spaces. 1966.

[227] Faezehossadat Khademi, Sayed Mohammadmehdi Jamal, Neela Deshpande, and Shreenivas N. Londhe. Predicting strength of recycled aggregate concrete using artificial neural network, adaptive neuro-fuzzy inference system and multiple linear regression. *International journal of sustainable built environment*, 5:355–369, 2016.

[228] Kabu Khadka and Soniya Maharjan. Customer satisfaction and customer loyalty. *Centria University of Applied Sciences Pietarsaari*, 1(10):58–64, 2017.

[229] H. E. Khalid, G. D. Gungor, and M. A. N. Zainal. Neutrosophic superhyper bi-topological spaces: Original notions and new insights. *Neutrosophic Sets and Systems*, 51:33–45, 2022.

[230] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[231] Huseyin Selcuk Kilic and Ahmet Selcuk Yalcin. Comparison of municipalities considering environmental sustainability via neutrosophic dematel based topsis. *Socio-economic Planning Sciences*, page 100827, 2020.

[232] Dan Klein and Christopher D Manning. Fast exact inference with a factored model for natural language parsing. *Advances in neural information processing systems*, 15, 2002.

[233] Puay Ping Koh, Yiik Diew Wong, and P. Chandrasekar. Safety evaluation of pedestrian behaviour and violations at signalised pedestrian crossings. *Safety Science*, 70:143–152, 2014.

[234] S Krishnaprakash, R Mariappan, and Said Broumi. Cubic spherical neutrosophic sets and selection of electric truck using cosine similarity measure. *Neutrosophic Sets and Systems*, 67(1):15, 2024.

[235] Cherukuri Aswani Kumar and Sunitha C. Srinivas. Concept lattice reduction using fuzzy k-means clustering. *Expert Syst. Appl.*, 37:2696–2704, 2010.

[236] Lakshya Kumar, Apurva Jindal, and Nagendra Rao Velaga. Financial risk assessment and modelling of ppp based indian highway infrastructure projects. *Transport Policy*, 62:2–11, 2017.

[237] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[238] M Lewis. Bart: Denoising sequence-to-sequence pre-training for natural language generation, translation, and comprehension. *arXiv preprint arXiv:1910.13461*, 2019.

[239] Chaojun Li, Kuo-Kuang Chang, and Shih-Ming Ou. Applying fuzzy statistics to evaluate and allocate resources in seven star grotto national park area. *IOP Conference Series: Materials Science and Engineering*, 1113, 2021.

[240] Dan Li, Jitender S. Deogun, William Spaulding, and Bill Shuart. Towards missing data imputation: A study of fuzzy k-means clustering method. In *Rough Sets and Current Trends in Computing*, 2004.

[241] Irene Li, Keen You, Xiangru Tang, Yujie Qiao, Lucas Huang, Chia-Chun Hsieh, Benjamin Rosand, and Dragomir R. Radev. Ehrkit: A python natural language processing toolkit for electronic health record texts. *ArXiv*, abs/2204.06604, 2022.

[242] Mark Junjie Li, Michael K. P. Ng, Yiu ming Cheung, and Joshua Zhexue Huang. Agglomerative fuzzy k-means clustering algorithm with selection of number of clusters. *IEEE Transactions on Knowledge and Data Engineering*, 20:1519–1534, 2008.

[243] Xia Li, Zhisheng Zhong, Jianlong Wu, Yibo Yang, Zhouchen Lin, and Hong Liu. Expectation-maximization attention networks for semantic segmentation. *2019 IEEE/CVF International Conference on Computer Vision (ICCV)*, pages 9166–9175, 2019.

[244] Xihua Li, Yun Luo, Hongya Wang, Jiong Lin, and Bin Deng. Doctor selection based on aspect-based sentiment analysis and neutrosophic topsis method. *Eng. Appl. Artif. Intell.*, 124:106599, 2023.

[245] Yanfeng Li, Hongzhong Huang, Jinhua Mi, Weiwen Peng, and X. M. Han. Reliability analysis of multi-state systems with common cause failures based on bayesian network and fuzzy probability. *Annals of Operations Research*, 311:195 – 209, 2019.

[246] Zheng Li, Yue Zhao, Xiyang Hu, Nicola Botta, Cezar Ionescu, and George H. Chen. Ecod: Unsupervised outlier detection using empirical cumulative distribution functions. *IEEE Transactions on Knowledge and Data Engineering*, 35:12181–12193, 2022.

[247] Thierry Libert. Models for a paraconsistent set theory. *J. Appl. Log.*, 3:15–41, 2005.

[248] Xiao Liu, Yanan Zheng, Zhengxiao Du, Ming Ding, Yujie Qian, Zhilin Yang, and Jie Tang. Gpt understands, too. *AI Open*, 5:208–215, 2024.

[249] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[250] Languang Lu, Xuebing Han, Jianqiu Li, Jianfeng Hua, and Minggao Ouyang. A review on the key issues for lithium-ion battery management in electric vehicles. *Journal of Power Sources*, 226:272–288, 2013.

[251] Minxia Luo and Guofeng Zhang. Divergence-based distance for picture fuzzy sets and its application to multi-attribute decision-making. *Soft Computing*, pages 1–17, 2023.

[252] M. M., M. Suneetha, Maria Mikhailova, Sripada Nsvsc Ramesh, and Kollati Vijaya Kumar. Leveraging double-valued neutrosophic set for real-time chronic kidney disease detection and classification. *International Journal of Neutrosophic Science*.

[253] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[254] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[255] Pabitra Kumar Maji. *Neutrosophic soft set*. Infinite Study, 2013.

[256] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[257] Christopher D Manning. *Foundations of statistical natural language processing*. The MIT Press, 1999.

[258] Christopher D Manning, Mihai Surdeanu, John Bauer, Jenny Rose Finkel, Steven Bethard, and David McClosky. The stanford corenlp natural language processing toolkit. In *Proceedings of 52nd annual meeting of the association for computational linguistics: system demonstrations*, pages 55–60, 2014.

[259] James G March. *A primer on decision making: How decisions happen*. Simon and Schuster, 1994.

[260] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.

[261] Mourad Oqla Massa'deh. On hyper q-fuzzy normal hx subgroup, conjugate and its.

[262] Manoj Mathew, Ripon Kumar Chakrabortty, and Michael J. Ryan. A novel approach integrating ahp and topsis under spherical fuzzy sets for advanced manufacturing system selection. *Eng. Appl. Artif. Intell.*, 96:103988, 2020.

[263] John T. Mentzer, William DeWitt, James S. Keebler, Soon Hong Min, Nancy W. Nix, Carlo D. Smith, and Zach G. Zacharia. Defining supply chain management. *Journal of Business Logistics*, 22:1–25, 2001.

[264] Irwin Miller. Probability, random variables, and stochastic processes. *Technometrics*, 8:378–380, 1966.

[265] Geltrude Mingrone, Simona Panunzi, Andrea De Gaetano, Caterina Guidone, Amerigo Iaconelli, Laura Leccesi, Giuseppe Nanni, Alfons Pomp, Marco Castagneto, Giovanni Ghirlanda, et al. Bariatric surgery versus conventional medical therapy for type 2 diabetes. *New England Journal of Medicine*, 366(17):1577–1585, 2012.

[266] Md Mofijur, I. M. Rizwanul Fattah, Md. Asraful Alam, A.K.M. Saiful Islam, Hwai Chyuan Ong, S. M. Ashrafur Rahman, Golamhassan Najafi, S.F. Ahmed, Md. Alhaz Uddin, and Teuku Meurah Indra Mahlia. Impact of covid-19 on the social, economic, environmental and energy domains: Lessons learnt from a global pandemic. *Sustainable Production and Consumption*, 26:343 – 359, 2020.

[267] G Mohanapriya, S Muthukumar, MM Shanmugapriya, et al. Kalman bucy filtered neuro fuzzy image denoising for medical image processing. *Neutrosophic Sets and Systems*, 70:314–330, 2024.

[268] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[269] Bernard Monjardet. On the use of ultrafilters in social choice theory. *Social choice and welfare*, 5:73–78, 1983.

[270] Sagvan Y Musa and Baravan A Asaad. A novel approach towards parameter reduction based on bipolar hypersoft set and its application to decision-making. *Neutrosophic Sets and Systems*, 55(1):33, 2023.

[271] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[272] Nada A Nabeeh, Florentin Smarandache, Mohamed Abdel-Basset, Haitham A El-Ghareeb, and Ahmed Aboelfetouh. An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. *Ieee Access*, 7:29734–29744, 2019.

[273] Nada A. Nabeeh, Florentin Smarandache, Mohamed Abdel-Basset, Haitham A. El-Ghareeb, and Ahmed Aboelfetouh. An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. *IEEE Access*, 7:29734–29744, 2019.

[274] Sorin Nădăban, Simona Dzitac, and Ioan Dzitac. Fuzzy topsis: a general view. *Procedia computer science*, 91:823–831, 2016.

[275] V Nayagam et al. A total ordering on n-valued refined neutrosophic sets using dictionary ranking based on total ordering on n-valued neutrosophic tuplets. *Neutrosophic Sets and Systems*, 58(1):23, 2023.

[276] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[277] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[278] Constantin Virgil Negoita and Anca L. Ralescu. Simulation, knowledge-based computing, and fuzzy statistics. *Technometrics*, 31:129–130, 1987.

[279] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[280] Robert W. Noyes, Arthur T. Hertig, and John Andrew Rock. Dating the endometrial biopsy. *American journal of obstetrics and gynecology*, 122 2:262–3, 1950.

[281] Serafim Opricovic and Gwo-Hshiung Tzeng. Compromise solution by mcdm methods: A comparative analysis of vikor and topsis. *Eur. J. Oper. Res.*, 156:445–455, 2004.

[282] Krzysztof Palczewski and Wojciech Sałabun. The fuzzy topsis applications in the last decade. *Procedia Computer Science*, 159:2294–2303, 2019.

[283] Dragan Pamucar, Morteza Yazdani, Radojko Obradovic, Anil Kumar, and Mercedes Torres-Jiménez. A novel fuzzy hybrid neutrosophic decision-making approach for the resilient supplier selection problem. *International Journal of Intelligent Systems*, 35(12):1934–1986, 2020.

[284] Athanasios Papoulis. Probability, random variables and stochastic processes. 1965.

[285] Athanasios Papoulis and S. Unnikrishna Pillai. Probability, random variables, and stochastic processes. 2002.

[286] Antonios Paraskevas and Michael Madas. A neutrosophic model for measuring evolution, involution, and indeterminacy in species: Integrating common and uncommon traits in environmental adaptation. *Neutrosophic Systems with Applications*, 23:23–32, 2024.

[287] Antonios Paraskevas and Florentin Smarandache. Modeling social evolution, involution, and indeterminacy: A neutrosophic-cultural algorithm approach. *Neutrosophic Sets and Systems*, 77:153–168, 2025.

[288] Vasile Patrascu. Penta and hexa valued representation of neutrosophic information. *arXiv preprint arXiv:1603.03729*, 2016.

[289] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[290] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[291] Zdzisław Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[292] Zdzisław Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[293] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[294] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[295] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[296] Xindong Peng and Yong Yang. Some results for pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 30, 2015.

[297] Zhang peng Tian, Hong yu Zhang, Jing Wang, Jian qiang Wang, and Xiao hong Chen. Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets. *International Journal of Systems Science*, 47:3598 – 3608, 2015.

[298] Arthur E Poropat. A meta-analysis of the five-factor model of personality and academic performance. *Psychological bulletin*, 135(2):322, 2009.

[299] John G. Proakis. Probability, random variables and stochastic processes. *IEEE Trans. Acoust. Speech Signal Process.*, 33:1637, 1985.

[300] Jhp Julwan Hendry Purba. Fuzzy probability on reliability study of nuclear power plant probabilistic safety assessment: A review. *Progress in Nuclear Energy*, 76:73–80, 2014.

[301] Peng Qi, Yuhao Zhang, Yuhui Zhang, Jason Bolton, and Christopher D. Manning. Stanza: A python natural language processing toolkit for many human languages. *ArXiv*, abs/2003.07082, 2020.

[302] Mohammad Naved Qureshi and Mohd Vasim Ahamad. An improved method for image segmentation using k-means clustering with neutrosophic logic. *Procedia computer science*, 132:534–540, 2018.

[303] Alec Radford. Improving language understanding by generative pre-training. 2018.

[304] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[305] George Rahonis. Fuzzy languages. In *Handbook of Weighted Automata*, pages 481–517. Springer, 2009.

[306] D. Rajan. Probability, random variables, and stochastic processes. 2017.

[307] Hao ran Lin, Bing yuan Cao, and Yun zhang Liao. Fuzzy statistics and fuzzy probability. 2018.

[308] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[309] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[310] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[311] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[312] V. Venkateswara Rao and Y. Srinivasa Rao. Neutrosophic pre-open sets and pre-closed sets in neutrosophic topology. *viXra*, 2017.

[313] Abdolreza Rashno, Florentin Smarandache, and Saeed Sadri. Refined neutrosophic sets in content-based image retrieval application. In *2017 10th Iranian Conference on Machine Vision and Image Processing (MVIP)*, pages 197–202. IEEE, 2017.

[314] Lisbeth J Reales-Chacón, María E Lucena de Ustáriz, Francisco J Ustáriz-Fajardo, Andrea C Peñafiel Luna, Gabriela J Bonilla-Ayala, Pablo Djabayan-Djibeyan, José L Erazo-Parra, and Mónica A Valdiviezo-Maygua. Study of the efficacy of neural mobilizations to improve sensory and functional responses of lower extremities in older adults with diabetic peripheral neuropathy using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 74:1–12, 2024.

[315] Stefano Crespi Reghizzi, Luca Breveglieri, and Angelo Morzenti. *Formal languages and compilation*. Springer, 2013.

[316] Ehud Reiter and Robert Dale. Building applied natural language generation systems. *Natural Language Engineering*, 3(1):57–87, 1997.

[317] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[318] Muhammad Riaz and Syeda Tayyba Tehrim. A robust extension of vikor method for bipolar fuzzy sets using connection numbers of spa theory based metric spaces. *Artificial Intelligence Review*, 54:561 – 591, 2020.

[319] Jesús Estupiñán Ricardo, Jorge Juan Domínguez Menéndez, Ignacio Fernando Barcos Arias, Jorge Manuel Macías Bermúdez, and Noel Moreno Lemus. Neutrosophic k-means for the analysis of earthquake data in ecuador. *Neutrosophic Sets & Systems*, 44, 2021.

[320] Robert J Richards. *The meaning of evolution: The morphological construction and ideological reconstruction of Darwin's theory*. University of Chicago Press, 2009.

[321] Michelle Richardson, Charles Abraham, and Rod Bond. Psychological correlates of university students' academic performance: a systematic review and meta-analysis. *Psychological bulletin*, 138(2):353, 2012.

[322] John T Rickard, Janet Aisbett, and Greg Gibbon. Fuzzy subsethood for fuzzy sets of type-2 and generalized type-$\{n\}$. *IEEE Transactions on Fuzzy Systems*, 17(1):50–60, 2008.

[323] Denis Riordan and Bjarne K Hansen. A fuzzy case-based system for weather prediction. *Engineering Intelligent Systems for Electrical Engineering and Communications*, 10(3):139–146, 2002.

[324] Rosa M Rodríguez, Luis Martínez, Vicenç Torra, ZS Xu, and Francisco Herrera. Hesitant fuzzy sets: state of the art and future directions. *International journal of intelligent systems*, 29(6):495–524, 2014.

[325] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[326] P Roopadevi, M Karpagadevi, S Krishnaprakash, Said Broumi, and S Gomathi. Comprehensive decision-making with spherical fermatean neutrosophic sets in structural engineering. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[327] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[328] Benjamin Rossman. Homomorphism preservation theorems. *J. ACM*, 55:15:1–15:53, 2008.

[329] Grzegorz Rozenberg and Arto Salomaa. *Handbook of Formal Languages: Volume 3 Beyond Words*. Springer Science & Business Media, 2012.

[330] Farnaz Sabahi and Mohammad Reza Akbarzadeh-Totonchi. Introducing validity in fuzzy probability for judicial decision-making. *Int. J. Approx. Reason.*, 55:1383–1403, 2014.

[331] Abhijit Saha, Irfan Deli, and Said Broumi. Hesitant triangular neutrosophic numbers and their applications to madm. *Neutrosophic Sets and Systems*, 35:269–298, 2020.

[332] Mehmet Sahin, Vakkas Ulucay, and Merve Meneke. Some new operations of (α,β,γ) interval cut set of interval valued neutrosophic sets. *Journal of Mathematical and Fundamental Sciences*, 2018.

[333] Rıdvan Şahin and Ahmet Küçük. On similarity and entropy of neutrosophic soft sets. *Journal of intelligent & fuzzy systems*, 27(5):2417–2430, 2014.

[334] Kavita Sahu, Fahad Ahmed Al-Zahrani, R. K. Srivastava, and Rajeev Kumar. Hesitant fuzzy sets based symmetrical model of decision-making for estimating the durability of web application. *Symmetry*, 12:1770, 2020.

[335] Rekha Sahu, Satya Ranjan Dash, and Sujit Das. Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory. 2021.

[336] A. A. Salama, Said Broumi, and Florentin Smarandache. Neutrosophic crisp open set and neutrosophic crisp continuity via neutrosophic crisp ideals. *International Journal of Information Engineering and Electronic Business*, 6:1–8, 2014.

[337] Mahmood M Salih, BB Zaidan, AA Zaidan, and Mohamed A Ahmed. Survey on fuzzy topsis state-of-the-art between 2007 and 2017. *Computers & Operations Research*, 104:207–227, 2019.

[338] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[339] Sabu Sebastian and TV Ramakrishnan. Multi-fuzzy sets. In *International Mathematical Forum*, volume 5, pages 2471–2476, 2010.

[340] Sabu Sebastian and TV Ramakrishnan. Multi-fuzzy sets: An extension of fuzzy sets. *Fuzzy Information and Engineering*, 3:35–43, 2011.

[341] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[342] Kristina P. Sinaga and Miin-Shen Yang. Unsupervised k-means clustering algorithm. *IEEE Access*, 8:80716–80727, 2020.

[343] Prem Kumar Singh. Turiyam set a fourth dimension data representation. *Journal of Applied Mathematics and Physics*, 9(7):1821–1828, 2021.

[344] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[345] Pritpal Singh and Surya Sekhar Bose. Ambiguous d-means fusion clustering algorithm based on ambiguous set theory: Special application in clustering of ct scan images of covid-19. *Knowledge-Based Systems*, 231:107432, 2021.

[346] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[347] Pushpinder Singh. Correlation coefficients for picture fuzzy sets. *J. Intell. Fuzzy Syst.*, 28:591–604, 2015.

[348] Joy K. Singhv and DeeptiKakkar. Source localization based healthcare diagnostic system using erp signals: An application to asd. 2021.

[349] Hans Slabbekoorn. Songs of the city: noise-dependent spectral plasticity in the acoustic phenotype of urban birds. *Animal Behaviour*, 85:1089–1099, 2013.

[350] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[351] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[352] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[353] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[354] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[355] Florentin Smarandache. *Definitions derived from neutrosophics*. Infinite Study, 2003.

[356] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[357] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[358] Florentin Smarandache. *Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability*. Infinite Study, 2013.

[359] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[360] Florentin Smarandache. Introduction to neutrosophic statistics. *ArXiv*, abs/1406.2000, 2014.

[361] Florentin Smarandache. Degrees of membership>1 and<0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[362] Florentin Smarandache. Neutrosophic overset. *Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics, Pons Editions Brussels*, 2016.

[363] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[364] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[365] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[366] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[367] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[368] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[369] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *ArXiv*, abs/1808.03948, 2018.

[370] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[371] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[372] Florentin Smarandache. *Introduction to neutrosophic genetics*. Infinite Study, 2020.

[373] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[374] Florentin Smarandache. Plithogenic probability & statistics are generalizations of multivariate probability & statistics. 2021.

[375] Florentin Smarandache. History of superhyperalgebra and neutrosophic superhyperalgebra (revisited again). *Neutrosophic Algebraic Structures and Their Applications*, page 10, 2022.

[376] Florentin Smarandache. Introducing a theory of neutrosophic evolution: Degrees of evolution, indeterminacy, and involution. *Collected Papers. Volume XIV: Neutrosophics and other topics*, page 159, 2022.

[377] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[378] Florentin Smarandache. Neutrosophic statistics is an extension of interval statistics, while plithogenic statistics is the most general form of statistics (second version). *International Journal of Neutrosophic Science*, 2022.

[379] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[380] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[381] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[382] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[383] Florentin Smarandache. New types of topologies and neutrosophic topologies. *Neutrosophic Systems with Applications*, 1:1–3, 2023.

[384] Florentin Smarandache. New types of topologies and neutrosophic topologies (improved version). *Neutrosophic Sets and Systems*, 57(1):14, 2023.

[385] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[386] Florentin Smarandache. Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions. 2024.

[387] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[388] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[389] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[390] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.

[391] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[392] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[393] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[394] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[395] Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.

[396] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[397] Eduardo Soares, Pyramo Costa Jr, Bruno Costa, and Daniel Leite. Ensemble of evolving data clouds and fuzzy models for weather time series prediction. *Applied Soft Computing*, 64:445–453, 2018.

[398] Le Hoang Son, Pham Van Viet, and Pham Van Hai. Picture inference system: a new fuzzy inference system on picture fuzzy set. *Applied Intelligence*, 46:652 – 669, 2016.

[399] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[400] Samir K. Srivastava. Green supply-chain management: A state-of-the-art literature review. *IO: Regulation*, 2007.

[401] Donald E Stanley and Daniel G Campos. The logic of medical diagnosis. *Perspectives in Biology and Medicine*, 56(2):300–315, 2013.

[402] V. S. Subha and R. Selvakumar. An innovative electric vehicle selection with multi-criteria decision-making approach in indian brands on a neutrosophic hypersoft rough set by using reduct and core. *2024*

Fourth International Conference on Advances in Electrical, Computing, Communication and Sustainable Technologies (ICAECT), pages 1–7, 2024.

[403] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[404] Siti Noraini Sulaiman and Nor Ashidi Mat Isa. Adaptive fuzzy-k-means clustering algorithm for image segmentation. *IEEE Transactions on Consumer Electronics*, 56, 2010.

[405] Chia-Chi Sun. A performance evaluation model by integrating fuzzy ahp and fuzzy topsis methods. *Expert systems with applications*, 37(12):7745–7754, 2010.

[406] Heng Sun and Wang Ni. Design and application of an ai-based text content moderation system. *Scientific Programming*, 2022.

[407] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[408] Seyed Mahmoud Taheri. Trends in fuzzy statistics. *Austrian Journal of Statistics*, 32:239–257, 2016.

[409] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[410] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[411] Valery V. Tuchin. Tissue optics: Light scattering methods and instruments for medical diagnosis. 2000.

[412] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[413] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[414] Vakkas Ulucay, Adil Kılıç, Memet Sahin, and Harun Deniz. *A new hybrid distance-based similarity measure for refined neutrosophic sets and its application in medical diagnosis*. Infinite Study, 2019.

[415] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[416] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[417] Dhanasingh Sivalinga Vijayan, Eugeniusz Koda, Arvindan Sivasuriyan, Johannes Winkler, Parthiban Devarajan, Ramamoorthy Sanjay Kumar, Aleksandra Jakimiuk, Piotr Osiski, Anna Podlasek, and Magdalena Daria Vaverková. Advancements in solar panel technology in civil engineering for revolutionizing renewable energy solutions-a review. *Energies*, 2023.

[418] Chia-Nan Wang, Ngoc-Ai-Thy Nguyen, and Thanh-Tuan Dang. Offshore wind power station (owps) site selection using a two-stage mcdm-based spherical fuzzy set approach. *Scientific Reports*, 12, 2022.

[419] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[420] Jingqian Wang, Xiaohong Zhang, and Qingqing Hu. Three-way fuzzy sets and their applications (ii). *Axioms*, 11(10):532, 2022.

[421] Meng Wang and Fanghui Hu. The application of nltk library for python natural language processing in corpus research. *Theory and Practice in Language Studies*, 2021.

[422] Tao Wang, Xinxing Wu, Harish Garg, Li-Yu Daisy Liu, and Guanrong Chen. A prospect theory-based mabac algorithm with novel similarity measures and interactional operations for picture fuzzy sets and its applications. *Eng. Appl. Artif. Intell.*, 126:106787, 2023.

[423] Zhongjie Wang and Minghe Lin. Finite element analysis method of slope stability based on fuzzy statistics. *Earth Sciences Research Journal*, 25:123–130, 2021.

[424] Zach Weber. Transfinite numbers in paraconsistent set theory. *The Review of Symbolic Logic*, 3:71 – 92, 2010.

[425] Zach Weber. Transfinite cardinals in paraconsistent set theory. *The Review of Symbolic Logic*, 5:269 – 293, 2012.

[426] Guiwu Wei and Hui Gao. The generalized dice similarity measures for picture fuzzy sets and their applications. *Informatica*, 29:107–124, 2018.

[427] Guiwu Wei and Yu Wei. Similarity measures of pythagorean fuzzy sets based on the cosine function and their applications. *International Journal of Intelligent Systems*, 33:634 – 652, 2018.

[428] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[429] Samuel Stanley Wilks. *Mathematical statistics*. Read Books Ltd, 2012.

[430] David Wilson. *Darwin's cathedral: Evolution, religion, and the nature of society*. University of Chicago press, 2019.

[431] Kim Witte and Mike Allen. A meta-analysis of fear appeals: Implications for effective public health campaigns. *Health Education & Behavior*, 27:591 – 615, 2000.

[432] Thomas Wolf, Lysandre Debut, Victor Sanh, Julien Chaumond, Clement Delangue, Anthony Moi, Pierric Cistac, Tim Rault, Rémi Louf, Morgan Funtowicz, et al. Transformers: State-of-the-art natural language processing. In *Proceedings of the 2020 conference on empirical methods in natural language processing: system demonstrations*, pages 38–45, 2020.

[433] Paul M Wortman. Medical diagnosis: An information-processing approach. *Computers and Biomedical Research*, 5(4):315–328, 1972.

[434] Weizhi Wu and Lei Zhou. On intuitionistic fuzzy topologies based on intuitionistic fuzzy reflexive and transitive relations. *Soft Computing*, 15:1183–1194, 2011.

[435] Yonghui Wu, Mike Schuster, Z. Chen, Quoc V. Le, Mohammad Norouzi, Wolfgang Macherey, Maxim Krikun, Yuan Cao, Qin Gao, Klaus Macherey, Jeff Klingner, Apurva Shah, Melvin Johnson, Xiaobing Liu, Lukasz Kaiser, Stephan Gouws, Yoshikiyo Kato, Taku Kudo, Hideto Kazawa, Keith Stevens, George Kurian, Nishant Patil, Wei Wang, Cliff Young, Jason R. Smith, Jason Riesa, Alex Rudnick, Oriol Vinyals, Gregory S. Corrado, Macduff Hughes, and Jeffrey Dean. Google's neural machine translation system: Bridging the gap between human and machine translation. *ArXiv*, abs/1609.08144, 2016.

[436] Jinglin Xu, Junwei Han, Kai Xiong, and Feiping Nie. Robust and sparse fuzzy k-means clustering. In *International Joint Conference on Artificial Intelligence*, 2016.

[437] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.

[438] Rui Yang, Qingcheng Zeng, Keen You, Yujie Qiao, Lucas Huang, Chia-Chun Hsieh, Benjamin Rosand, Jeremy Goldwasser, Amisha D. Dave, Tiarnán D. L. Keenan, Emily Y. Chew, Dragomir R. Radev, Zhiyong Lu, Hua Xu, Qingyu Chen, and Irene Li. Ascle: A python natural language processing toolkit for medical text generation. 2023.

[439] Xibei Yang, Dongjun Yu, Jingyu Yang, and Chen Wu. Generalization of soft set theory: from crisp to fuzzy case. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*, pages 345–354. Springer, 2007.

[440] Yong Yang, Xia Tan, and Congcong Meng. The multi-fuzzy soft set and its application in decision making. *Applied Mathematical Modelling*, 37(7):4915–4923, 2013.

[441] Zhilin Yang. Xlnet: Generalized autoregressive pretraining for language understanding. *arXiv preprint arXiv:1906.08237*, 2019.

[442] Jun Ye and Shigui Du. Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship. *International Journal of Machine Learning and Cybernetics*, 10:347 – 355, 2017.

[443] Hasan Yeti? and Mehmet Karakose. Modelling type-2 fuzzy systems by optimized nonstationary fuzzy sets with genetic algorithm. *2020 24th International Conference on Information Technology (IT)*, pages 1–4, 2020.

[444] Jeonghee Yi, Tetsuya Nasukawa, Razvan Bunescu, and Wayne Niblack. Sentiment analyzer: Extracting sentiments about a given topic using natural language processing techniques. In *Third IEEE international conference on data mining*, pages 427–434. IEEE, 2003.

[445] Adem Yolcu and Taha Yasin Ozturk. Fuzzy hypersoft sets and it's application to decision-making. *Theory and application of hypersoft set*, 50, 2021.

[446] Keun-Young Yoo. Cancer control activities in the republic of korea. *Japanese journal of clinical oncology*, 38 5:327–33, 2008.

[447] Hong yu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[448] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[449] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[450] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[451] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[452] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[453] Lemnaouar Zedam, Naeem Jan, Ewa Rak, Tahir Mahmood, and Kifayat Ullah. An approach towards decision-making and shortest path problems based on t-spherical fuzzy information. *International Journal of Fuzzy Systems*, 22:1521 – 1534, 2020.

[454] Aiwu Zhang, Jianguo Du, and Hongjun Guan. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. *J. Intell. Fuzzy Syst.*, 29:2697–2706, 2015.

[455] Dayong Zhang, Minwen Hu, and Qiang Ji. Financial markets under the global pandemic of covid-19. *Finance Research Letters*, 36:101528 – 101528, 2020.

[456] Hongyu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615 – 627, 2015.

[457] Qi Zhang, Chunjie Zhou, Yu-Chu Tian, Naixue N. Xiong, Yuanqing Qin, and Bowen Hu. A fuzzy probability bayesian network approach for dynamic cybersecurity risk assessment in industrial control systems. *IEEE Transactions on Industrial Informatics*, 14:2497–2506, 2018.

[458] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

[459] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[460] Wen-Ran Zhang. Bipolar fuzzy sets. 1997.

[461] Wen-Ran Zhang. Yinyang bipolar fuzzy sets and fuzzy equilibrium relations: for clustering, optimization, and global regulation. *Int. J. Inf. Technol. Decis. Mak.*, 5:19–46, 2006.

[462] Xiaolu Zhang and Zeshui Xu. Extension of topsis to multiple criteria decision making with pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 29, 2014.

[463] Hu Zhao and Hong-Ying Zhang. On hesitant neutrosophic rough set over two universes and its application. *Artificial Intelligence Review*, 53:4387–4406, 2020.

[464] Xuerong Zhao and Bao Qing Hu. Fuzzy and interval-valued fuzzy decision-theoretic rough set approaches based on fuzzy probability measure. *Inf. Sci.*, 298:534–554, 2015.

[465] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.